Distributed Fiedler Vector Estimation with Application to Desynchronization of Harmonic Oscillator Networks

Diego Deplano, Mauro Franceschelli, Alessandro Giua and Luca Scardovi

***Abstract*—The Fiedler vector of a graph is the eigenvector corresponding to the the algebraic connectivity, which is the second-smallest eigenvalue (counting multiple eigenvalues separately) of the corresponding Laplacian matrix. We propose a continuous-time distributed control protocol to drive the value of the state variables of a network toward the Fiedler vector, up to a scale factor. Our protocol is unbiased and robust with respect to the initial network state, but the knowledge of the algebraic connectivity is required. By means of the proposed control law, we design a local state feedback that achieves desynchronization on arbitrary undirected connected networks of diffusively coupled harmonic oscillators. We provide numerical simulations to corroborate the theoretical results.**

***Index Terms*—Fiedler, eigenvector, distributed estimation, desynchronization, harmonic oscillators, networks.**

I. Introduction

COMPUTING the eigenvectors of a graph Laplacian is a problem of fundamental importance for various applications and it is the cornerstone of spectral graph theory [1]. Among all eigenvectors, the *Fiedler vector* [2] plays a pivotal role: it is the eigenvector corresponding to the second smallest eigenvalue (counting multiple eigenvalues separately) of the Laplacian matrix, also known as the *algebraic connectivity*. The Fiedler vector is useful in graph partitioning [3], [4], [5] and in the control of algebraic connectivity [6], [7], [8].

Power Iteration (PI) [9] is an established iterative method to compute the leading eigenvalue(s) and the associated eigenvector(s) of a matrix. In [6], [8], [10], [11], [12] the Fiedler vector is computed by means of methods based on a distributed implementation of PI. Main drawbacks of [6], which exploits the algorithm proposed in [13], are the centralized initialization step and the high number and size of the messages the nodes need to exchange. In [8] and [11] the decentralization is carried out at each agent by two consensus estimators, which are required to run "fast enough" in order to expect the resulting algorithm to converge. Similar approaches are used to compute Laplacian eigenvalues including the algebraic connectivity [14]. Another class of algorithms forces the nodes to oscillate at eigenfrequencies and deduce spectral information through Fast Forurier Transform (FFT). In [15] the Fiedler vector is computed by running at every node the wave equation and computing the eigenvector components through an FFT. This algorithm is proved to be orders of magnitude faster than PI-based algorithms. An FFT approach for distributedly computing the eigenvalues is given in [16].

On one hand, the main limitation of PI-based approaches consists on the distributed normalization of the vectors at each step, which severely affects their convergence speed and requires a centralized initialization step. On the other hand, FFT-based approaches suffer from a rather poor accuracy and robustness issues.

The first contribution of this paper is a continuous-time distributed control protocol to drive a MAS toward the Fiedler vector of its graph Laplacian. The proposed protocol relies neither on a distributed PI nor a FFT approach and it ensures high convergence speed, accuracy and robustness to initial conditions. However, it assumes the knowledge of the algebraic connectivity. Since various distributed algorithms have been proposed to estimate all the eigenvalues of an undirected graph's Laplacian [17], [16], [18], [19], such an assumption is plausible for static networks, as in the case of our main application.

The second contribution is to exploit the proposed protocol to design a local feedback law to desynchronize a network of coupled harmonic oscillators. While synchronization has been formally defined [20], [21], [22] as a configuration maximizing the order-parameter (magnitude of the centroid of the oscillators), the opposite definition of *desynchronization* is more ambiguous [23], [24], [25], [26]. In this work, we define desynchronization as a configuration zeroing the order-parameter, which is dual to the classical definition of synchronization given in [20]. Desynchronized configurations correspond to network states with zero mean: such is the case of the Fiedler vector.

The paper is structured as follows. After introducing notation and preliminaries in Section II, a distributed protocol in continuous-time is proposed and employed in Section III to distributedly estimate the Fiedler vector in single-integrator MASs and in Section IV to achieve desynchronization in networks of diffusively coupled harmonic oscillators. In Section V simulations corroborating the theoretical results are provided. Concluding remarks are given in Section VI.

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II. Preliminaries

We adopt the following notation. The sets $\mathbb{R}$ and $\mathbb{R}_+$ denote, respectively, the reals and the nonnegative reals. The set $\mathbb{S}^1$ denotes the *unit circle*. The set $\mathbb{C}$ denotes the set of the complex numbers. Given n vectors $x_1, x_2, \dots, x_n \in \mathbb{R}^m$ we indicate with $x = [x_1^T, x_2^T, \dots, x_n^T]^T \in \mathbb{R}^{nm}$ the stacking of the vectors. We denote with I_n and $\mathbf{1}_n$ the identity matrix and a vector of ones of dimension n; subscripts are omitted if the dimension is clear from the context. With $A \otimes B$ we denote the Kronecker product of two matrices A and B of opportune dimensions. Given a complex number number $w \in \mathbb{C}$, its real and imaginary part are denoted $\Re\{w\}$ and $\Im\{w\}$, respectively.

A. Multi Agent Systems

Given a Multi-Agent System (MAS), the pattern of interactions among the agents is encoded by an undirected *graph*. A graph $\mathcal{G} = (V, E)$ consists of a set of nodes $V = \{1, \dots, n\}$ representing the agents, and a set of edges $E \subseteq \{V \times V\}$. An edge $(i, j) \in E$, with $i \neq j$, means that i and j can communicate. To each agent i is associated a set of neighbors $\mathcal{N}_i = \{j \in V : (i, j) \in E\}$, representing the set of agents communicating with agent i. Since the graph is undirected, $(i, j) \in E$ if and only if $(j, i) \in E$. An undirected graph is said to be *connected* if between any pair of nodes $i, j \in V$ there exists a *path*, i.e., a finite sequence of adjacent edges that connects node i to node j. The adjacency matrix $A = \{a_{ij}\}$ of a graph $\mathcal{G}$ is an $n \times n$ matrix with coefficients

$$a_{ij} := \begin{cases} 1 & \text{for } (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The degree of each node is defined as $d_i = \sum_{i=1}^{n} a_{ij}$. The degree matrix is defined as $D = \text{diag}(d)$, and the Laplacian matrix of the graph is defined as $L = D - A$. Let $\lambda_{L,i}$ and $v_{L,i}$ be the eigenvalues and corresponding eigenvectors of the Laplacian matrix L. The eigenvalues $\lambda_{L,i}$ are real and satisfy $0 = \lambda_{L,1} \leq \lambda_{L,2} \leq \dots \leq \lambda_{L,n}$. In particular, $\lambda_{L,2} > 0$ if and only if the graph is connected. The *algebraic connectivity* is denoted as $\lambda_{L,2}$ and its associated eigenvector $v_{L,2}$, known as the *Fiedler vector*, satisfies $\mathbf{1}^T v_{L,2} = 0$ if $\lambda_2 > 0$.

III. Fiedler vector Estimation

Consider a network of n single integrators

$$\dot{y}_i = u_i, \qquad \forall i = 1, \dots, n, \tag{1}$$

where $y_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are, respectively, the state and the input of agent i. The network's topology is represented by an undirected graph $\mathcal{G} = (V, E)$ and the control law u_i is said to be local if each agent can only exchange information with its neighbors. The goal of this section to design a local control law such that each agent i estimates the i-th Fiedler vector component of the Laplacian matrix L of $\mathcal{G}$.

For the sake of thoroughness, we first provide a global control law from which a local version is derived in the next subsection.

Theorem 1: Consider a MAS with agents dynamics (1) driven by the control law

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} (y_j - y_i) - \alpha \mathbf{1}^T y + \lambda_{L,2} y_i. \tag{2}$$

If $\mathcal{G}$ is connected and $\alpha > \lambda_{L,2}/n$ then the MAS converges to a state proportional to the Fiedler vector of graph G.

Proof: The dynamics can be written in compact form as

$$\dot{y} = -\underbrace{\left[L + \alpha \mathbf{1}\mathbf{1}^T - \lambda_{L,2} I\right]}_{M} x.$$

The construction of matrix M consists of two conceptual steps [27]. 1) *Matrix inflation*: add a term $\alpha \mathbf{1}\mathbf{1}^T$ to the Laplacian L with $\alpha > \lambda_{L,2}/n$. The smallest eigenvalue of the resulting matrix is $\lambda_{M,1} = \lambda_{L,2}$ while the eigenvectors associated to the eigenvalues remain unchanged. 2) *Eigenvalues shifting*: subtract matrix $\lambda_{L,2} I$ to the resulting matrix. All eigenvalues are shifted by $\lambda_{L,2}$ thus ensuring that $\lambda_{M,1} = 0$ is a single null eigenvalue. It follows that matrix M has spectrum $0 = \lambda_{M,1} < \lambda_{M,2} \leq \dots \leq \lambda_{M,n}$ with $v_{M,1} = v_{L,2}$. Thus, the MAS is marginally stable and y converges asymptotically to the Fiedler vector, completing the proof. ∎

A. Main Result

The control law (2) is not local as it relies on two global information: the algebraic connectivity λ_2 and an average of the state variables $\mathbf{1}^T y(t)$. In this work we make the standing assumption that the algebraic connectivity λ_2 is known. Such an assumption is plausible for static networks since it can be inferred in a distributed way by several algorithms [17], [16], [18], [19]. On the other hand, the problem of computing in real time the average $\mathbf{1}^T y(t)$ can be overcome by employing a distributed estimator of this quantity, as proposed in earlier work [28], [29], [30]. In particular, we consider an *integral dynamic consensus algorithm*, which differs from those presented in [30].

Theorem 2: Consider a MAS with agents dynamics (1) driven by the control law

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} (y_j - y_i) - \alpha v_i + \lambda_{L,2} y_i. \tag{3}$$

where v_i is given by

$$\begin{aligned} \dot{v}_i &= \dot{y}_i + \beta(y_i - v_i) + K_I \sum_{j \in \mathcal{N}_i} a_{ij} (z_j - z_i), \\ \dot{z}_i &= K_I v_i. \end{aligned} \tag{4}$$

If $\mathcal{G}$ is connected, $\alpha > \lambda_{L,2}$, $\beta > 0$, $K_I > 0$, then $y(t)$ converges almost globally to a state proportional to the Fiedler vector of graph $\mathcal{G}$.

Proof: Let $w = \begin{bmatrix} y & v & z \end{bmatrix}^T \in \mathbb{R}^{3n}$. The dynamics can be written in compact form as

$$\dot{w} = \underbrace{\begin{bmatrix} \lambda_{L,2} I - L & -\alpha I & \mathbf{0} \\ (\lambda_{L,2} + \beta) I - L & -(\alpha + \beta) I & -K_I L \\ \mathbf{0} & K_I I & \mathbf{0} \end{bmatrix}}_{M} w.$$

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To compute the eigenvalues of matrix M, we proceed by solving $\det\{M - \lambda I_{3n}\} = 0$. Let us partition matrix $M - \lambda I_{3n} = [A\ B; C\ D]$ where

$$A = \begin{bmatrix} (\lambda_{L,2} - \lambda)I - L & -\alpha I \\ (\lambda_{L,2} + \beta)I - L & -(\alpha + \beta + \lambda)I \end{bmatrix},$$
$$B = \begin{bmatrix} \mathbf{0} \\ -K_I L \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{0} & K_I I \end{bmatrix}, \quad D = -\lambda I.$$

By Shur-complement, the determinant of $M - \lambda I_{3n}$ is equal to $\det\{D\} \cdot \det\{A - BD^{-1}C\}$, where $\det\{D\} = \lambda^n$ and $\det\{A - BD^{-1}C\} = \det\{a_1 L^2 + a_2 L + a_3 I\}$, with

$$a_1 = \frac{K_I^2}{\lambda}, \qquad a_2 = K_I^2 - \lambda_{L,2}\frac{K_I^2}{\lambda} + \beta + \lambda,$$
$$a_3 = \lambda\ \alpha + \lambda\ \beta + \lambda^2 - \lambda_{L,2}\ \beta - \lambda_{L,2}\ \lambda + \alpha\beta.$$

Finally, letting $H = a_1 L + a_2 I$, we can write

$$\det\{M - \lambda I_{3n}\} = \lambda^n \cdot \det\{HL + a_3 I\} = 0. \tag{5}$$

Matrix HL is a product of two commuting matrices, i.e., $HL = LH$, thus any eigenvalue of HL is a product of the eigenvalues of H and L. Furthermore, H and L share the same set of eigenvectors because matrix $H = a_1 L + a_2 I$ where a_1 and a_2 are real scalars. Thus, from (5) one can derive the next relationships

$$\lambda(\lambda_{L,i}\lambda_{H,i} + a_3) = 0, \qquad \forall i \in V,$$

where the eigenvalues $\lambda_{L,i}$ are real and satisfy $0 = \lambda_{L,1} < \lambda_{L,2} \leq \cdots \leq \lambda_{L,n}$ since the graph is assumed to be connected. The eigenvalues of H are $\lambda_{H,i} = a_1\ \lambda_{L,i} + a_2$, thus, substituting coefficients a_i yields for any $\forall i \in V$

$$\lambda^3 + b_i\lambda^2 + c_i\lambda + d_i = 0, \quad \begin{array}{l} b_i = \alpha + \beta - \lambda_{L,2} + \lambda_{L,i} \\ c_i = (\alpha - \lambda_{L,2})\beta + (K_I^2 + \beta)\lambda_{L,i} \\ d_i = K_I^2\lambda_{L,i}(\lambda_{L,i} - \lambda_{L,2}) \end{array}.$$

One can notice that M has three eigenvalues for each $\lambda_{L,i}$ with $i \in V$, which can be computed by the above equation. We proceed by ensuring that all eigenvalues of matrix M have negative real part using the Routh criteria.

For $i = \{1, 2\}$ it holds $d_i = 0$ and, by the Routh criteria specialized for second degree polynomials, all solutions are strictly negative if and only if coefficients b_i, c_i are positive, which is verified if the conditions of the theorem hold.

For $i \in \{3, \ldots, n\}$ it holds $d_i > 0$ and, by the Routh criteria specialized for third degree polynomials, all solutions are strictly negative if and only if in addition it holds $b_i c_i - d_i > 0$, proved next

$$b_i c_i - d_i = \beta(\lambda_{L,2} - \lambda_{L,i})^2 + (2\alpha\beta + \beta^2)(\lambda_{L,i} - \lambda_{L,2}) + \\ + (\alpha\beta + K_I^2\lambda_{L,i})(\alpha + \beta) > 0,$$

since $\lambda_{L,i} \geq \lambda_{L,2}$ for $i \in \{3, \ldots, n\}$.

We conclude that, under the conditions of the theorem, matrix M has two zero eigenvalues because $d_i = 0$ for $i \in \{1, 2\}$ while all other eigenvalues have negative real part. Stability of the system can be ensured by proving that the geometric multiplicity of eigenvalue 0 is equal to its algebraic multiplicity, which is two. We prove this fact by showing that two distinct eigenvectors are associated to the zero eigenvalue. Recalling that $w = \begin{bmatrix} y & v & z \end{bmatrix}^T \in \mathbb{R}^{3n}$ is the state of the overall system, we compute $\dot{w} = Mw = \mathbf{0}$,

$$\begin{cases} (\lambda_{L,2}I - L)y = \mathbf{0} \\ \beta y - K_I L z = \mathbf{0} \\ v = \mathbf{0} \end{cases},$$

There are two feasible choices for y. First choice is $y = \mathbf{0}$, leading to $Lz = \mathbf{0}$, i.e., $z = \delta v_{L,2}$, $\forall\ \delta \in \mathbb{R}$. Second choice is $y = \delta v_{L,2}$, leading to $z = \frac{\beta\delta}{K_I\lambda_{L,2}}v_{L,2} + \sigma\ \mathbf{1}$, $\forall\ \sigma \in \mathbb{R}$. Thus, the zero eigenvalue has the two distinct eigenvectors

$$e_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ v_{L,2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} v_{L,2} \\ \mathbf{0} \\ \frac{\beta}{K_I\lambda_{L,2}}v_{L,2} + \frac{\sigma}{\delta}\ \mathbf{1} \end{bmatrix}. \tag{6}$$

We conclude that the system is marginally stable, and

$$\lim_{t\to\infty} y(t) = \gamma v_{L,2}, \qquad \gamma \in \mathbb{R}.$$

Let $S \subset \mathbb{R}^{3n}$ be the space orthogonal to e_2. Coefficient γ is null if and only if $w(0) \in S$, which is a set of measure zero with respect to $\mathbb{R}^2$. Thus, if $w(0) \notin S$ then $y(t)$ almost globally converges to the Fiedler vector, completing the proof. ■

Remark 1: Theorem 2 ensures that

$$y(t) = \lim_{t\to\infty} \gamma v_{L,2}$$

with $\gamma \neq 0$ if and only if the initial condition of the system is not orthogonal to the eigenvector e_2 given in (6). In the vicinity of this critical hyperplane of initial conditions, γ can be small thus leading to a large numerical error in the estimation of the Fiedler vector.

IV. Desynchronization in Harmonic Oscillator Networks

In this section we define and study the desynchronization problem in networks of n coupled identical harmonic oscillators with natural frequency $\omega \in \mathbb{R}$. A harmonic oscillator is a second-order linear system characterized by both amplitude $M(t)$ and phase $\theta(t)$ as opposed to the popular Kuramoto oscillator, a first-order non-linear model, which describes only the phase. We consider a network of controlled harmonic oscillators, where the position $p_i(t) = M_i(t)\cos(\omega t + \theta_i(t)) \in \mathbb{R}$ of the i-th oscillators has the following dynamics, see [21], [22]

$$\ddot{p}_i + \omega^2 p_i = u_i^c + u_i^d, \tag{7}$$

where $u_i^c \in \mathbb{R}$ is the local control feedback to be designed, and

$$u_i^d = \sum_{j\in\mathcal{N}_i} a_{ij}\left(\dot{p}_j - \dot{p}_i\right), \tag{8}$$

accounts for the diffusive coupling between the oscillators. The coefficients $a_{ij} \in \{0, 1\}$ are the entries of the adjacency matrix of an undirected graph $\mathcal{G}$ describing both the coupling and the communication networks: $a_{ij} = 1$ if oscillator i is coupled and can communicate with oscillator j, and $a_{ij} = 0$ otherwise. Introducing the state vector $x_i = [p_i \quad \dot{p}_i]^T \in \mathbb{R}^2$,

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the state-space representation of a network of oscillators (7) can be written as

$$\begin{aligned} \dot{x}_i &= Ax_i + B(u_i^c + u_i^d), \qquad i \in V, \\ y_i &= Cx_i, \end{aligned} \tag{9}$$

where the state matrices are given by

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \qquad C = B^T = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Suppose that the steady state output of each oscillator i is

$$y_i^{ss}(t) = M_i \cos(\omega t + \ \theta_i) = \Re\left\{ M_i e^{j\theta_i} \ \cdot e^{j\omega t} \right\},$$

where $M_i \in \mathbb{R}_+$, $\theta_i \in \mathbb{S}^1$ are the steady state magnitude and the phase of oscillator i and j denotes the imaginary unit. Thus, the collective steady-state output dynamics

$$\mathbf{1}^T y^{ss}(t) = \ \sum_{i=1}^{n} y_i^{ss}(t) = \ \Re\left\{ \sum_{i=1}^{n} M_i e^{j\theta_i} \cdot e^{j\omega t} \right\} \tag{10}$$

is related to the centroid[1]

$$Re^{j\phi} = \ \frac{1}{\sum_{i=1}^{n} M_i} \sum_{i=1}^{n} M_i e^{j\theta_i}, \qquad \sum_{i=1}^{n} M_i > 0, \tag{11}$$

where $R \in \mathbb{R}_+$ represents the phase-coherence of the population of oscillators and $\phi \in \mathbb{S}^1$ indicates the average phase. The goal is to derive a local feedback to achieve desynchronization in a network of diffusively coupled harmonic oscillators in the sense shown next.

Definition 1 (Desynchronization measure): Consider a network of n identical oscillators (9). The network is said to achieve *desynchronization* if

$$R = 0 \ \Leftrightarrow \ \mathbf{1}^T y^{ss}(t) = 0, \tag{12}$$

i.e., the collective steady-state output dynamics (10) is non-null with zero mean or, equivalently, the centroid (11) is at the origin of the Complex Plane. ■

In the light of the above definition, consider a simple yet illustrative example of a network of three oscillators. Fig. 1 depicts the following configurations: (a) all phase differences are null, then oscillators are not desynchronized, regardless of their amplitude; (b) amplitudes are equal and the phase differences are $\theta_1 - \theta_2 = \frac{2\pi}{3}$, $\theta_2 - \theta_3 = \frac{2\pi}{3}$, then the oscillators are desynchronized, this configuration is referred in the literature as a *splay state*; (c) if $M_1 = 2M_2 = 2M_3$ and $\theta_2 = \theta_3 = \theta_1 + \pi$, then the oscillators are not in a splay state but they are desynchronized.

[1]If all oscillators have the same fixed amplitude (as in the case for Kuramoto oscillators) the centroid reduces to $Re^{j\phi} = \frac{1}{n}\sum_{i=1}^{n} e^{j\theta_i}$ and R is known as the *order parameter*.

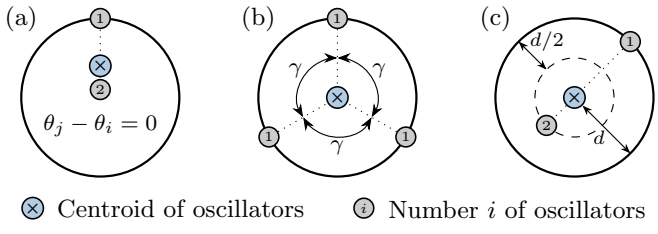


Fig. 1. Steady-state configurations of a network of three oscillators: (a) not desynchronization, (b) desynchronization with same amplitudes, (c) desynchronization with different amplitudes.

A. Main Result

It is known that a network of identical harmonic oscillators (9) under the diffusive coupling (8) achieves synchronization if the interconnection graph is symmetric [31], [32]. The feedback provided in the next theorem is able to cancel out the synchronization effect of the diffusive coupling while asymptotically steering the network to a non-trivial desynchronized state.

Theorem 3 (Desynchronization of harmonic oscillators): Consider a network of n identical harmonic oscillators (9) coupled with the diffusive coupling (8) and driven by the control law

$$u_i = -\ \beta v_i + \ \lambda_{L,2} y_i. \tag{13}$$

where v_i is given by (4). If $\mathcal{G}$ is connected,

$$\alpha > \max\left\{ \ \lambda_{L,2}, \frac{\omega^2}{2\lambda_{L,2}} \right\}, \quad K_I = \ \sqrt{\frac{2\alpha^2 + \ \omega^2}{2\lambda_{L,2}}}, \tag{14}$$

then the network achieves desynchronization almost globally.

Proof: Let $w_i = \ \begin{bmatrix} x_i & v_i & z_i \end{bmatrix}^T \in \mathbb{R}^4$. The network of coupled harmonic oscillators (9) subject to the diffusive coupling (8) and the feedback control (3) can be written in compact form as

$$\dot{w} = \overbrace{\Big[(I \otimes A^*) - (L \otimes B^*) \Big]}^{M} w \tag{15}$$

$$y = (I \otimes C^*) w, \tag{16}$$

where the operator $\otimes$ denotes the Kronecker product and

$$A^* = \begin{bmatrix} A + \lambda_{L,2} BC & -\alpha B & \mathbf{0} \\ C(A + (\lambda_{L,2} + \alpha) I_2) & -2\alpha & 0 \\ \mathbf{0} & K_I & 0 \end{bmatrix},$$

$$B^* = \begin{bmatrix} BC & \mathbf{0} & \mathbf{0} \\ C & 0 & K_I \\ \mathbf{0} & 0 & 0 \end{bmatrix}, \quad C^* = \begin{bmatrix} C & 0 & 0 \end{bmatrix}.$$

Since L is symmetric, there exists an orthogonal matrix P such that $\Lambda = P^T L P$ is a diagonal matrix. Consider the coordinate change

$$\widetilde{w} = \underbrace{\Big[P \otimes I_4 \Big]}_{\widetilde{P}} w \quad \rightarrow \quad \dot{\widetilde{w}} = \underbrace{\Big[\widetilde{P}^T M \ \widetilde{P} \Big]}_{\widetilde{M}} \widetilde{w}. \tag{17}$$

where, by exploiting the properties of the Kronecker product,

$$\widetilde{M} = \ \Big[(I \otimes A^*) - (\Lambda \otimes B^*) \Big].$$

Matrices M and $\widetilde{M}$ share the same spectrum. Matrix $\widetilde{M}$ is a block diagonal matrix with blocks $\widetilde{M}_i$ given by

$$\widetilde{M}_i = A^* - \lambda_{L,i} B^* \qquad \forall i \ \in V,$$

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where the eigenvalues $\lambda_{L,i}$ of L are real and satisfy $0 = \lambda_{L,1} \leq \lambda_{L,2} \leq \cdots \leq \lambda_{L,n}$ since the graph is assumed to be connected. It is known that the eigenvalues of the block diagonal matrix $\widetilde{M}$ are the eigenvalues of the blocks $\widetilde{M}_i$. By means of the Routh criterion, it can be shown (we omit the steps due to space constraints) that under condition (14) all the eigenvalues of blocks $\widetilde{M}_i$ are strictly inside the left half of the complex plane, except for block $\widetilde{M}_1$, which has a zero eigenvalue, and block $\widetilde{M}_2$, which has a pair of imaginary conjugate eigenvalues. Exploiting the change of variable (17), one can write the state evolution of the system as

$$w(t) = e^{Mt}w(0) = e^{\widetilde{P}\widetilde{M}\widetilde{P}^T t}w(0) = \widetilde{P}e^{\widetilde{M}t}\widetilde{P}^T w(0).$$

and so the output

$$y(t) = (I_n \otimes C^*)w(t) = (I_n \otimes C^*)\widetilde{P}e^{\widetilde{M}t}\widetilde{P}^T w(0).$$

As $t \to \infty$, all blocks $e^{\widetilde{M}_i t}$ for $i = 3, \ldots, n$ tend to zero because of the negative real part of their eigenvalues. Thus, since the columns of P are the eigenvectors of matrix L it follows

$$\begin{aligned}\lim_{t\to\infty} y(t) &= (I_n \otimes C^*)\left[v_{L,1}v_{L,1}^T \otimes e^{\widetilde{M}_1 t} + v_{L,2}v_{L,2}^T \otimes e^{\widetilde{M}_2 t}\right]w(0) \\ &= \left[(v_{L,1}v_{L,1}^T \otimes C^* e^{\widetilde{M}_1 t}) + (v_{L,2}v_{L,2}^T \otimes C^* e^{\widetilde{M}_2 t})\right]w(0).\end{aligned}$$

Matrix $\widetilde{M}_1$ has only one null eigenvalue (the others have negative real part) with eigenvector $\mathbf{1}_5 \otimes [0\ 0\ 0\ 1]^T$, thus

$$\lim_{t\to\infty} e^{\widetilde{M}_1 t} \propto \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1\end{bmatrix}.$$

Since $\lim_{t\to\infty} C^* e^{\widetilde{M}_1 t} = \mathbf{0}$ it follows

$$\lim_{t\to\infty} y(t) = \gamma\left[v_{L,2}v_{L,2}^T \otimes C^* e^{\widetilde{M}_2 t}\right]w(0),$$

with $\gamma \neq 0$ if and only if $w(0)$ is not orthogonal to the eigenvector corresponding to the null eigenvalue. The collective steady-state output dynamics results in

$$\mathbf{1}^T y^{ss}(t) = \lim_{t\to\infty} \mathbf{1}^T y(t) = \left[\mathbf{0} \otimes C^* e^{\widetilde{M}_2 t}\right]w(0) = \mathbf{0}$$

We proved that the network reaches a steady-state with a zero mean output, in which each oscillator has a non-trivial oscillatory behaviour due to the pair of imaginary conjugate eigenvalues of block $\widetilde{M}_2$, and so of M. ■

V. Numerical Results

A. Fiedler vector estimation

We compare our protocol with the one in [8] because of their similar structure. We keep our notation for any common parameter (i.e., α, β and K_I) while we use the notation in [8] for the remaining parameters (i.e., k_2, k_3 and K_P). Since our algorithm benefits from the knowledge of the algebraic connectivity, to ensure a fair comparison the state variables common to both algorithms have been initialized to the same value, while the remaining state variables in [8], which are related to a dynamic consensus estimator, are chosen such that the initial estimation error is null.

In the first simulation, a MAS with $n = 5$ agents (1), random topology and local control law (3) is considered. Accordingly with conditions of Theorem 2, we choose the gains for the integral dynamic average consensus estimator as $\beta = 25$, $K_I = 10$, and the gain for the eigenvector estimator as $\alpha = 6$. The additional gains for the algorithm in [8] are chosen as $k_2 = 1$, $k_3 = 20$ and $K_P = 50$. Network topology and free parameters are the same of Example 1 in [8].

In the second simulation, we consider a MAS with increasing number n of agents, line topology and local control law (3). With the choice of the line topology we are considering the worst case scenario for the dynamic average estimator. According to conditions of Theorem 2, the gains for the integral dynamic average consensus estimator are chosen $\beta = 10$, $K_I = 15$, the gain for the eigenvector estimator is $\alpha = 2\lambda_{L,2}$. The additional gains for the algorithm in [8] are chosen as $k_2 = 1$, $k_3 = 2\lambda_{L,2}$ and $K_P = 25$.

The Figures 2-3 show the results of the two simulations just described. Fig 2 shows the error evolution $e(t) = ||y(t) - \widetilde{v}_{L,2}||$, $\widetilde{v}_{L,2} = \lim_{t\to\infty} ||y(t)||v_{L,2}$. Fig. 3 shows, for different values of n, the time required by the two algorithms to reach an error of the order of 10^{-6}. These simulations show that the additional knowledge of the algebraic connectivity, which is required by the proposed approach, can help to achieve a faster convergence rate and make the algorithm scalable with the number of the agents in the network.

B. Desynchronization of harmonic oscillators

Synchronization of networked mechanical oscillator systems have been subject of interest [21], [22], [33], [34]. Here we give a physical example as an application of Theorem 3.

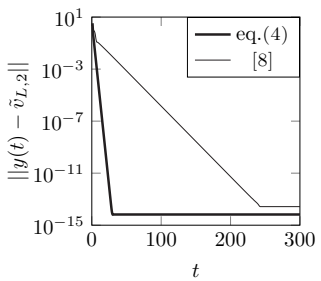


Fig. 2. Fiedler vector estimation error in a network of 5 agents.

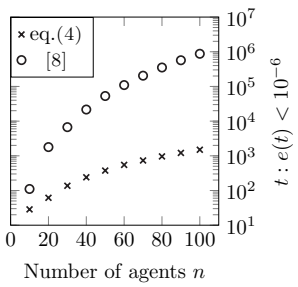


Fig. 3. Convergence time of Fiedler vector estimation in line networks.

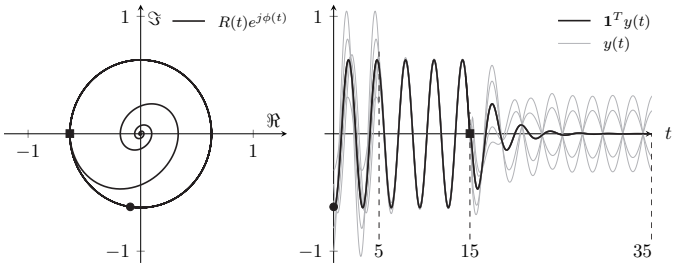


Fig. 4. Evolution of 5 harmonic oscillators: the initial state is denoted with a circle. The diffusive coupling (8) is activated at $t = 5$ and the desynchronizing feedback (13)-(4) is activated at $t = 15$, denoted with a square.

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Consider the networked mechanical systems consisting of n train wagons of identical mass m with linear dumper b interaction [34] and a mass-spring-damper modeling the interaction with the ground. Assuming a null damping in order to guarantee the best comfort to the passengers, and denoting the spring coefficient with k, the model becomes the one in (7)-(8) with natural frequency $\omega = \sqrt{k/m} = 0.1$ and dumping coefficient $b = 1$. The feedback control (13) models the active suspensions between wagons and desynchronization corresponds to the minimum stress on the rails since the sum of the forces becomes zero as the time passes.

Simulation of a line-topology network with $n = 5$ nodes is shown in Fig. 4 with $\alpha = 1.88$ and $K_I = 2$, according to Theorem 3. The oscillators start evolving without any coupling until at $t = 5$ the diffusive coupling is enabled and synchronization is achieved. Let us denote with $R(t)e^{j\phi(t)}$ the centroid as defined in (10)-(11) given $x(t)$ as the initial condition of the network. Thus, as it can be seen in Fig. 4, the collective output dynamics has constant module $R(t) = 0.6$ for $t \in [0, 15)$. At $t = 15$ the proposed control feedback is activated and the network is shown to reach desynchronization in the sense of Definition 1, i.e., $R = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} \mathbf{1}^T y(t) = \mathbf{1}^T y^{ss}(t) = 0$.

VI. Conclusions

In this work we proposed a protocol for solving the problem of distributed Fiedler vector estimation in networks of single-integrator agents. Exploiting the zero mean property of the Fiedler vector, we used the proposed protocol to design a local feedback law to achieve desynchronization in a network of diffusively coupled harmonic oscillators. The main advantages of the proposed protocol are its robustness to re-initialization and a fast convergence rate. Future work will focus on relaxing the condition that the algebraic connectivity of the graph must be known, thus paving the way to address time-varying topologies.

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