Dynamic Consensus on the Median Value in Open Multi-Agent Systems

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***Abstract*—In this paper we consider a dynamic consensus problem in continuous time where the state variables of the agents track with zero error the median value of a set of time-varying reference signals given as input to the agents in a time-varying, undirected network topology. Then, we consider the performance of the protocol in the framework of open multi-agent systems by proposing join and leave mechanisms, i.e., the scenario where agents may join and leave the network during the protocol execution. We characterize the finite-time convergence properties and tracking error of the considered protocol in the case of inputs with bounded variations. One notable feature of consensus on the median value is the robustness of the median, as opposed to the average, with respect to abnormal or outlier values of inputs which represent the outcome of a measurement or estimation process, thus significantly increasing the robustness of the estimation for large scale networks. We use non-smooth Lyapunov theory to provide convergence guarantees and simple tuning rules to adjust the algorithm parameters.**

I. Introduction

During the last decade, the problem of consensus in multi-agent systems has been studied with special emphasis on graph theoretical methods. Consensus can be regarded as a control objective in which all the agents in a network converge to (or "agree upon") a common value. This is achieved through a given control strategy usually referred to as consensus algorithm. In this paper the focus is on *dynamic* consensus, namely, consensus on locally time-varying reference signals which is further complicated by considering an *open* scenario where agents may join and leave the network during the protocol execution.

A recent development in control and systems theory considers open-multi-agent systems as networks of coupled dynamical systems that can login and logout from the network, thus change the number of state variables during the transient and steady-state behavior of the multi-agent system. Recent works of the still limited literature on the topic can be found in [1], [2], [3], [4], [5].

The literature on dynamic consensus is, instead, rich. One of the early works on the topic is [6], which uses Laplacian feedback and provides as inputs to the agents the derivative of a set of reference signals. The requirement on precise initialization of the algorithm makes it not robust to initialization errors or changes in the network composition. To guarantee robustness with respect to initial errors, [7] and [8] proposed alternative dynamic average consensus algorithms whose convergence is guaranteed provided that the graph representing the network topology is connected, static and balanced.

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Other notable mentions in the consensus literature are [9] and [10] (the so called *ratio consensus*) to solve consensus on unbalanced digraphs by exploiting the ratio of two state variables that perform linear state updates with appropriate, distinct, initialization.

The method in this manuscript is based on a discontinuous local interaction rule. We refer to [11],[12],[13] for an exhaustive tutorial on how to study discontinuous gradient flows and discontinuous feedback systems by means of non-smooth Lyapunov theory.

In this paper, we present an algorithm that converges in finite time. Approaches that obtain consensus in finite time can be found in [14],[15],[16],[17],[18] and [19]. In particular, in [14],[15] and [16] the proposed approaches are capable of tracking with zero error the average value of time-varying smooth reference signals with bounded derivatives, in finite time on static undirected graph topologies. The dynamic average consensus protocol in [14] gurantees perfect tracking in finite time using one-hope communication, but it suffers from the limitation of being not robust to initial conditions. The protocol in [15] converges to zero error in finite time and is also robust to initial conditions, however it requires two-hop communication. The protocol in [16] uses one-hop communication, but the error synchronizes to zero exponentially, rather than in finite time. In [20] using continuous local interactions, finite time consensus is studied for single integrators and second order systems with unknown non-linear dynamics.

It is worth noting that, most averaging approaches in large scale networks suffer from a considerable limitation: the presence of even a single outlier value within the considered data may affect the network behavior. This is a strong motivation to pursue consensus strategies on robust statistics such as the median as opposed to the average.

In [21] and [22], to overcome the drawbacks of average consensus, the authors proposed algorithms to solve the consensus on the median value of a set of numbers. In more detail, in [22], consensus on the median value is achieved in finite time under static communication graphs and it is shown that the considered algorithm is resilient with respect to both outliers and uncooperative agents if a graph theortical condition based on k-connectivity is satisfied. In [21], a robust distributed consensus on the median value is proposed in the presence of matching perturbations to the agents dynamics. In contrast to this paper we now consider a scenario dynamic consensus on the median value in the framework of open multi-agent systems.

Summarizing, the **main contributions** of this paper are as follows: i) We characterize the performance of a variant of the local interaction rule in [22] to solve a dynamic consensus problem on the median value. ii) We consider the case in which the network topology is time-varying and connected at all times as opposed to time-invariant. iii) We show that if the reference signals of the agents have bounded derivative, even if the median value is not unique due to their even number, then the average of median values is continuous and is tracked in finite time by the agents. iv) We

consider an open-multi-agent scenario where agents join and leave the network, and characterize the convergence properties of the method. v) We show that in the open scenario the time-varying median value is discontinuous and the approach is able to track it with bounded error.

This work is organized as follows. In Section II we present preliminary notions and definitions. In Section III we introduce the dynamic consensus algorithm. In Section IV we characterize the convergence properties of the algorithm in both dynamic and open scenarios by exploiting non-smooth Lyapunov tools. In Section V we provide numerical simulations. Finally in Section VI concluding remarks and future perspectives are discussed.

II. Preliminaries

Let $\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t))$ be a connected time-varying undirected graph, where $\mathcal{V}(t)$ is a time-varying set of nodes (agents) and $\mathcal{E}(t) \subseteq \{\mathcal{V}(t) \times \mathcal{V}(t)\}$ is the set of edges. There exists an edge between agents i and j if $(i,j) \in \mathcal{E}$. Let $\mathcal{N}_i(t) = \{j \in \mathcal{V}(t) : (i,j) \in \mathcal{E}\}$ be the set of neighbors of agent i, i.e., agents that share an edge with agent i.

We define function "sign" and the set-valued function "SIGN" for $y \in \mathbb{R}$ as follows

$$\text{sign}(y) = \begin{cases} 1, \text{ if } y > 0, \\ 0, \text{ if } y = 0, \\ -1, \text{ if } y < 1, \end{cases} \quad \text{SIGN}(y) \in \begin{cases} 1, \text{ if } y > 0, \\ [-1,1], \text{ if } y = 0, \\ -1, \text{ if } y < 0. \end{cases} \tag{1}$$

Consider the dynamical system

$$\dot{x}(t) = f(x(t)), \qquad x(t) \in \mathbb{R}^n, \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{2}$$

where $f(x(t)) : \mathbb{R}^n \to \mathbb{R}^n$, is defined almost everywhere, i.e., it is defined for every $x(t) \in \mathbb{R}^n \setminus W$, where W is a subset of $\mathbb{R}^n$ of measure zero. Furthermore, $f(x(t))$ is measurable in an open region $Q \subset \mathbb{R}^n$ and for all compact sets $D \subset Q$ there exists a constant A_D such that $\|f(x(t))\| \leq A_D$ almost everywhere in D.

If the differential equation (2) has discontinuous right-hand side, following [23] we understand the corresponding solution in the so-called *Filippov sense* as the solution of an appropriate differential inclusion, as explained in the next definition.

***Definition** 2.1 (Filippov solution):* A vector function $x(\cdot) \in \mathbb{R}^n$ is called a Filippov solution of (2) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and, for almost all $t \in [t_0, t_1]$, satisfies the differential inclusion

$$\dot{x} \in K(x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} co\{f(B(x,\delta) \setminus N, t)\}, \tag{3}$$

where $\bigcap_{\mu(N)=0}$ denotes the intersection over all sets N of Lebesgue measure zero, $co\{\cdot\}$ denotes the convex hull and $B(x,\delta)$ is a ball of radius δ centered at x. ■

If $f(x)$ is measurable and locally bounded then the corresponding set-valued map is upper semicontinuous, compact, convex valued and locally bounded so that the differential inclusion (3) possesses a Filippov solution for each initial condition x_0. The reader is referred to [12] for a comprehensive tutorial on the topic.

We now recall the definition of *Clarke's Generalized Gradient* [24].

***Definition** 2.2 (Clarke's Generalized Gradient):* Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient $\partial V(x)$ is defined as

$$\partial V(x) \triangleq co\left\{\lim_{i \to \infty} \nabla V(x_i) | x_i \to x, x_i \notin \Omega_V \cup N\right\},$$

where ∇V denotes the conventional gradient, $x_i \in \mathbb{R}^n$ represents a point of an infinite succession which converges to $x \in \mathbb{R}^n$ as i grows to infinity, Ω_V is a set of Lebesgue measure zero which contains all points where $\nabla V(x)$ does not exist, and N is an arbitrary set of measure zero. ■

The Clarke's generalized gradient coincides with the standard gradient at the points where the standard derivative of the scalar function exists. Further details and examples of computation can be found in [11], [12], [24]. Next, we recall the definition of set-valued Lie derivative.

***Definition** 2.3:* Given a locally Lipschitz function $V(x)$, where $x \in \mathbb{R}^n$ is governed by the differential inclusion $\dot{x} \in K(x)$, the set-valued Lie derivative of $V(x)$ at x is

$$\widetilde{\mathcal{L}}V(x) = \{a \in \mathbb{R} | \exists v \in K(x) \text{ such that } \zeta \cdot v = a, \forall \zeta \in \partial V(x)\}. \tag{4}$$

The set-valued Lie derivative allows the study of the evolution of a Lyapunov function along the Filippov solutions of the dynamical system according to the next theorem.

***Theorem** 2.4 (Evolution along Filippov solutions [24]): Let $x(t) : [t_0, t_1] \to \mathbb{R}^n$ be a Filippov solution of* (3)*. Let $V(x(t))$ be a locally Lipschitz and regular function. Then $\frac{d}{dt}(V(x(t)))$ exists a.e. and $\frac{d}{dt}(V(x(t))) \in \widetilde{\mathcal{L}}V(x(t))$ a.e..* ■

A generalization of the extended Lyapunov theorem for non-smooth analysis appropriate for the study of consensus problem is the following ([11],[22]).

***Theorem** 2.5: Let $M = span(\mathbf{1}_n)$ be the subspace spanned by vector $\mathbf{1}_n$. Consider a scalar function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$, with $V(x(t)) = 0 \ \forall x \in M$ and $V(x(t)) > 0 \ \ \forall x \notin M$. Let $x(t) : \mathbb{R} \to \mathbb{R}^n$ and $V(x(t))$ be absolutely continuous on $[t_0, \infty)$ with $\frac{d}{dt}(V(x(t))) \leq -\epsilon < 0$ a.e. on $\{t | x(t) \notin M\}$. Then, $V(x(t))$ converges to 0 in finite time and $x(t)$ reaches the subspace M in finite time as well.* ■

III. Dynamic Consensus on the Median Value

We consider the local interaction protocol

$$\dot{x}_i(t) = -\lambda \sum_{j \in \mathcal{N}_i(t)} \text{sign}(x_i(t) - x_j(t)) - \alpha \, \text{sign}(x_i(t) - u_i(t)), \tag{5}$$

where $x_i(t) \in \mathbb{R}$ is the state of the i-th agent and $u_i(t)$ is a local reference signal given as input, which represents a measurement or more generaly an exogenous time-varying information.

When an agent joins the network at time t it holds

$$x_i(t) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} x_j(t). \tag{6}$$

When an agent leaves the network, its state variable and incident edges are removed from the system.

The convergence properties of (5) will be shown to be independent from initial conditions, as opposed to some dynamic consensus algorithms which are initialization dependent. Let us compactly define $u(t) = [u_1(t), \ldots, u_n(t)]$, as the vector set of reference signals at time t.

In this paper we characterize the convergence properties of the local interaction protocol in (5) with respect to the time-varying

median value of vector $u(t)$ and, in the second part of the paper, in the framework of open-multi-agent systems, i.e., the case where agents may join and leave the network as time goes by.

As first step in the analysis, we note that a Filippov solution to (5) exists for every initial condition of (5) because the corresponding right-hand side is uniformly bounded [12].

The median value of a set of real numbers is a number which is greater than half of the numbers in the set and smaller than the other half. The median value is not uniquely defined when the cradinality of the set is even.

***Definition** 3.1:* Let

$$\chi(t) = \arg\min_{\bar{u}(t)\in\mathcal{U}(t)} \sum_{j\in\mathcal{V}(t)} |u_j(t) - \bar{u}(t)|, \tag{7}$$

where $\mathcal{U}(t)$ is the set of all entries of vector $u(t)$. If the number of agents at time t is odd, then the solution of (7), i.e., the median value, is unique and set $\chi(t)$ is a singleton. On the other hand, if at time t the number of agents is even, then $\chi(t)$ contains two solutions, both with minimum objective value. By averaging the elements of set $\chi(t)$ we define the (unique) m-value $m(u(t))$ of vector $u(t)$ as

$$m(u(t)) = \frac{1}{|\chi(t)|} \sum_{\xi_r\in\chi(t)} \xi_r, \tag{8}$$

where $|\chi(t)|$ denotes the cardinality of the set. We also define the so-called median interval $\mathcal{M}(u(t))$ as

$$\mathcal{M}(u(t)) = \left[\min_{\xi_r\in\chi(t)} \xi_r, \max_{\xi_r\in\chi(t)} \xi_r\right]. \tag{9}$$

Any number inside the median interval $\mathcal{M}(u(t))$ is a median value.

We also define by $L_M(t)$ the time-varying length of the median interval

$$L_M(t) = \max_{\xi_r\in\chi(t)} \xi_r - \min_{\xi_r\in\chi(t)} \xi_r. \tag{10}$$

Clearly $L_M(t) = 0$ if the number of agents is odd at time t. ■

In this paper we make the following assumption which is common in the dynamic consensus literature.

***Assumption** 3.2:* All reference signals have bounded derivative, namely

$$\exists \quad \Pi \in \mathbb{R}^+ : \qquad |\dot{u}_i(t)| \le \Pi \qquad \forall i \in \mathcal{V}(t). \tag{11}$$

■

From the above assumption it directly follows the next result.

***Proposition** 3.3:* If Assumption 3.2 holds, then

$$|\dot{m}(u(t))| \le \Pi. \tag{12}$$

Proof: Let $\Upsilon(t) = \{i \in \mathcal{V}(t) : u_i(t) = \xi_r,\ \xi_r \in \chi(t)\}$. By eq. (8) it follows that $\dot{m}(u(t)) = \frac{1}{|\Upsilon(t)|}\sum_{i\in\Upsilon(t)} \dot{u}_i(t)$, and by Assumption 3.2

$$|\dot{m}(u(t))| = \left|\frac{1}{|\Upsilon(t)|} \sum_{i\in\Upsilon(t)} \dot{u}_i(t)\right| \le \frac{1}{|\Upsilon(t)|} \sum_{i\in\Upsilon(t)} \Pi = \Pi. \tag{13}$$

■

We also assume that the input signals satisfy the following assumption.

***Assumption** 3.4:* There exists $B \in \mathbb{R}^+$ such that, for all $t \ge 0$, it holds

$$\max_{i\in\mathcal{V}(t)} (u_i(t)) - \min_{i\in\mathcal{V}(t)} (u_i(t)) \le B. \tag{14}$$

■

Furthermore, we consider the following three assumptions on the network topology and on the way it may change.

***Assumption** 3.5:* Graph $\mathcal{G}(t)$ is undirected and connected for all $t \ge 0$ and the set of instants of time in which $\mathcal{G}(t)$ changes is of measure zero. ■

***Assumption** 3.6:* There exists a minimium dwell time $\Delta\tau$ separating two consecutive instants in which agents join or leave the network. ■

***Assumption** 3.7:* There exists $n_{max} \in (0,\infty) : |\mathcal{V}(t)| \le n_{max},\ \forall t \ge 0$. ■

Finally, we formalize the definition of finite-time consensus.

***Definition** 3.8:* The state variables $x_i(t) \in \mathbb{R}$, $i \in \mathcal{V}(t)$, of a networked multi-agent system are said to reach a *finite-time consensus* if there exists a real value $T \in (0,\infty)$ and a real function $c(t) : \mathbb{R} \to \mathbb{R}$ such that

$$x_i(t) = c(t), \quad \forall i \in \mathcal{V}(t), \quad \forall t \ge T. \tag{15}$$

Function $c(t)$ is the time-varying consensus function. ■

IV. Finite-time convergence properties

In this section we discuss the convergence properties of the states of a set of agents following protocol (5). In particular we want to show that they converge in finite-time to the median value of the reference signals. To this aim we first prove that the agent's states reach consensus in finite-time. In particular, the time to reach consensus is a function of the parameters α and λ in protocol (5).

***Theorem** 4.1: Consider the network dynamics* (5) *along with a time-varying graph $\mathcal{G}(t)$ satifying Assumption 3.5. Agents may join or leave the network according to Assumptions 3.6 and 3.7. Let the tuning parameters satisfy $0 < \alpha < \frac{2\lambda}{n_{max}}$. Then, the consensus condition* (15) *is achieved in a finite time T upper bounded as:*

$$T \le T_1 = \frac{\max\limits_{i\in\mathcal{V}(t)} x_i(0) - \min\limits_{i\in\mathcal{V}(t)} x_i(0)}{\mu^2},\ \ \mu^2 = 2\left(\frac{2\lambda}{n_{max}} - \alpha\right). \tag{16}$$

Proof: Here, we extend the proof in [22] to account for a switching network topology, dynamic inputs and a scenario of open multi-agent systems. First, define the sets

$$I_{max}(t) = \{k \in \mathcal{V}(t) : x_k = \max_{i\in\mathcal{V}(t)} x_i(t)\},$$
$$I_{min}(t) = \{k \in \mathcal{V}(t) : x_k = \min_{i\in\mathcal{V}(t)} x_i(t)\},$$

and consider the non-smooth Lyapunov function

$$V_1(x(t)) = \frac{\sum\limits_{i\in I_{max}(t)} x_i(t)}{|I_{max}(t)|} - \frac{\sum\limits_{i\in I_{min}(t)} x_i(t)}{|I_{min}(t)|}. \tag{17}$$

By Assumptions 3.5 and 3.6, the instants of time in which graph $\mathcal{G}(t)$ changes define a set of measure zero, i.e., there can be an infinite but countable number of such instants of time. Therefore, since the derivatives of the state variables in (5) are bounded for almost all t, it follows that the state trajectories of (5) are absolutely continuous. Thus, also the sets $I_{max}(t)$ and $I_{min}(t)$ may change cardinality only at isolated instants of time. It follows that these time instants can be disregarded in the non-smooth Lyapunov analysis while evaluating the generalized gradient of $V_1(x(t))$.

Then, by applying the definition of Clarke's generalized gradient on the non-smooth Lyapunov function, the set-valued generalized time-derivative $\frac{d}{dt}\left(V_1(x(t))\right)$ can be computed following the steps in [22] in the time intervals in which $\mathcal{G}(t)$ is static and the set of agents does not change. In particular, it follows that in such time intervals it is

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq -\mu^2 \tag{18}$$

where μ^2 is the strictly positive constant computed in [22] and equal to $\mu^2 = 2\left(\frac{2\lambda}{n} - \alpha\right)$. The above estimate straightforwardly yields the finite-time convergence of $V_1(x(t))$ to zero, which in turns implies that the finite time consensus condition (16) is achieved.

Now, we show that when the set of agents changes, the value of the considered non-smooth Lyapunov function (17) does not change.

Let one agent join the network at time $t^+ = t + \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. According to (6), it joins with a state value equal to the average of its neighbours, i.e., inside the convex hull spanned by their states. Thus, we can write

$$\max_{i\in\mathcal{V}(t^+)} x_i(t) = \max_{i\in\mathcal{V}(t)} x_i(t), \qquad \min_{i\in\mathcal{V}(t^+)} x_i(t) = \min_{i\in\mathcal{V}(t)} x_i(t)$$

Therefore, the value of the Lyapunov function (17) does not change, i.e., $V_1(x(t^+)) = V_1(x(t))$. If, instead, an agent leaves the network, it is straightforward to verify that the Lyapunov function may decrease in value and $V_1(x(t^+)) \leq V_1(x(t))$. Now, by integrating the generalized time derivative (18) in an interval of time from which we remove all sets of instants of measure zero correponding to changes in the network topology, or agents that join or leave the network, and considering that when these events occur they do not increase the value of function $V_1(x(t))$, it holds

$$V_1(x(t)) = V_1(x(0)) + \int_0^t \frac{d}{dt}\left(V_1(x(t))\right) dt \leq V_1(x(0)) - \mu^2 t. \tag{19}$$

In [22] the parameter μ^2 was estimated to be $\mu^2 = 2\left(\frac{2\lambda}{n} - \alpha\right)$. In this paper the number of agents is time-varying, therefore we extend the same result by estimating its value as $\mu^2 = 2(\frac{2\lambda}{n_{max}} - \alpha)$ based on Assumption 3.7. Thus, by Theorem 2.5, $V_1(x(t))$ converges to zero in finite time and from (19) we can compute a maximum finite transient time which is equal to $T_1 = \left(\max_{i\in\mathcal{V}(t)} x_i(0) - \min_{i\in\mathcal{V}(t)} x_i(0)\right)/\mu^2$. ∎

The next theorem proves that the dynamic consensus function $c(t)$ converges in finite time and tracks with zero error the median value $m(u(t))$ despite a switching network topology if no agent joins or leaves the network.

***Theorem** 4.2: Consider the network dynamics* (5) *along with a time-varying graph $\mathcal{G}(t)$ with n agents satisfying Assumption 3.5. Let the tuning parameters, under Assumption 3.2, satisfy $n\Pi < \alpha < \frac{2\lambda}{n}$ and let the consensus condition* (15) *be in force starting from time $t = T$. Then, there exist $T_2 \geq T$ such that*

$$c(t) \in \mathcal{M}(u(t)), \quad \forall t \geq T_2, \tag{20}$$

where $\mathcal{M}(u(t))$ denotes the time-varying median interval which contains the median value (8) *of the set of reference signals and*

$$T_2 \leq \frac{|c(T) - m(u(T))| + L_M(T_2)}{\frac{\alpha}{n} - \Pi} + T. \tag{21}$$

Proof: By assumption, the consensus condition (15) is in force for all $t \geq T$. Consider the Lyapunov function

$$V_2(x(t), u(t)) = |c(t) - m(u(t))|, \quad t \geq T. \tag{22}$$

In the following, to simplify our notation we omit the dependence of $V_2(x(t), u(t))$ from $x(t)$ and $u(t)$ and refer to it as $V_2(t)$ instead. It is worth to note that due to (15) all agents hold the same state value $c(t)$, and therefore the average state value is given by $c(t)$, i.e.,

$$c(t) = \frac{\sum_{i\in\mathcal{V}} x_i(t)}{n}, \quad t \geq T. \tag{23}$$

The generalized gradient of V_2 takes the form

$$\partial V_2(t) = \mathrm{SIGN}(c(t) - m(u(t))). \tag{24}$$

By (23), the time varying consensus function $c(t)$ obeys the discontinuous differential equation $\dot{c}(t) = \frac{\sum_{i\in\mathcal{V}} \dot{x}_i(t)}{n}$, thus

$$\dot{c}(t) = -\frac{\displaystyle\sum_{i\in\mathcal{V}} \alpha\ \mathrm{sign}\left(x_i(t) - u_i(t)\right) + \lambda \sum_{j\in\mathcal{N}_i(t)} \mathrm{sign}\left(x_i(t) - x_j(t)\right)}{n}. \tag{25}$$

Due to the symmetry of local interactions resulting from the undirected nature of the considered graph $\mathcal{G}(t)$, it is $\sum_{i\in\mathcal{V}} \lambda\left(\sum_{j\in\mathcal{N}_i(t)} \mathrm{sign}\left(x_i(t) - x_j(t)\right)\right) = 0$, therefore (25) straightforwardly simplifies as

$$\dot{c}(t) = -\frac{\displaystyle\sum_{i\in\mathcal{V}} \alpha \cdot\ \mathrm{sign}\left(x_i(t) - u_i(t)\right)}{n}. \tag{26}$$

The Filippov solutions of (26) are governed by the differential inclusion [24]

$$\dot{c}(t) \in -\frac{1}{n}\alpha \sum_{i\in\mathcal{V}} \mathrm{SIGN}\left(x_i(t) - u_i(t)\right). \tag{27}$$

Let $I_{up}(t) = \{k \in \mathcal{V} : x_k(t) < u_k(t)\}$, $I_{down}(t) = \{k \in \mathcal{V} : x_k(t) > u_k(t)\}$, $I_{equal}(t) = \{k \in \mathcal{V} : x_k(t) = u_k(t)\}$. Clearly, the sets $I_{up}(t)$, $I_{down}(t)$ and $I_{equal}(t)$ are disjoint, and their union forms the set $\mathcal{V}$, thus

$$|I_{up}(t)| + |I_{down}(t|) + |I_{equal}(t)| = |\mathcal{V}| = n \tag{28}$$

By definition, the next relations hold

$$\mathrm{SIGN}\left(x_i(t) - u_i(t)\right) = -1, \qquad \forall i \in I_{up}(t), \tag{29}$$

$$\mathrm{SIGN}\left(x_i(t) - u_i(t)\right) = 1, \qquad \forall i \in I_{down}(t). \tag{30}$$

Therefore, by (29) and (30) one manipulates (27) as follows

$$\dot{c}(t) \in -\frac{\alpha}{n}\Bigg(|I_{down}(t)| - |I_{up}(t)| + \sum_{i\in I_{equal}(t)} \mathrm{SIGN}(x_i(t) - u_i(t))\Bigg). \tag{31}$$

The set-valued Lie derivative of $V_2(t)$ takes the form

$$\widetilde{\mathcal{L}}V_2(t) = \mathrm{SIGN}(c(t) - m(u(t)))\dot{c}(t) - \mathrm{SIGN}(c(t) - m(u(t)))\dot{m}(u(t)). \tag{32}$$

We notice that, by assumption, it is

$$x_i(t) = c(t), \qquad \forall i \in \mathcal{V}, \quad \forall t \geq T, \tag{33}$$

and by definition, $c(t) < u_i(t),\ \forall i \in I_{up}(t)$ and $c(t) > u_i(t),\ \forall i \in I_{down}(t)$. Having in mind the definition of the median value, the next implication holds

$$|I_{down}(t)| = |I_{up}(t)| \quad \Longrightarrow \quad c(t) \in \mathcal{M}(u(t)). \tag{34}$$

Therefore, we concentrate on the case in which $|I_{down}(t)| \neq |I_{up}(t)|$. When $|I_{down}(t)| \neq |I_{up}(t)|$, two cases may occur

$$\text{Case 1}: \left||I_{down}(t)| - |I_{up(t)}|\right| > |I_{equal}(t)|, \tag{35}$$

$$\text{Case 2}: ||I_{down}(t)| - |I_{up}(t)|| \leq |I_{equal}(t)|, \tag{36}$$

which will be treated separately.

Case 1. When relation (35) is in force, it holds

$$\sum_{i \in I_{equal}(t)} \alpha \ \text{SIGN}(x_i(t) - u_i(t)) \in [-\alpha |I_{equal}(t)|, \alpha |I_{equal}(t)|]. \tag{37}$$

Without loss of generality we consider the case $|I_{down}(t)| > |I_{up}(t)|$ and thus inequality (35) becomes $|I_{down}(t)| - |I_{up}(t)| > |I_{equal}(t)|$ (the same derivation can be carried out for the case $|I_{down}(t)| < |I_{up}(t)|$). Then, we consider the value of each term in eq. (31) thus yielding the next estimate

$$\begin{aligned} \alpha \left| |I_{down}(t)| - |I_{up}(t)| + \sum_{i \in I_{equal}(t)} \text{SIGN}(x_i(t) - u_i(t)) \right| \\ \geq \alpha \left(|I_{down}(t)| - |I_{up}(t)| - |I_{equal}(t)| \right). \end{aligned} \tag{38}$$

Denote

$$k(t) = |I_{up}(t)| + |I_{equal}(t)|. \tag{39}$$

By substituting eq. (39) into eq. (28), it follows that

$$|I_{down}(t)| = n - k(t). \tag{40}$$

Now substituting (39) into $|I_{down}(t)| > |I_{up}(t)| + |I_{equal}(t)|$, it holds that $|I_{down}(t)| > k(t)$. Since $|I_{down}(t)|$ and $k(t)$ are integer numbers, then $|I_{down}(t)| \geq k(t) + 1$. Thus, since the smaller lower bound of $|I_{down}(t)|$ occurs for n odd, it holds

$$k(t) \leq \frac{n-1}{2}. \tag{41}$$

In light of eq. (39) and eq. (40), the right hand side of eq. (38) can be rewritten as

$$\alpha \left(|I_{down}(t)| - |I_{up}(t)| - |I_{equal}(t)| \right) = \alpha \left(n - 2k(t) \right). \tag{42}$$

Since the right hand side of eq. (42) is a decreasing function of $k(t)$, its minimum subject to the constraint in eq. (41) is obtained when $k(t) = \frac{n-1}{2}$. Thus, it holds $\alpha \left(n - 2k(t) \right) \geq \alpha(n - 2\frac{n-1}{2}) = \alpha$ and by virtue of (38), one derives that

$$\alpha \left| |I_{down}(t)| - |I_{up}(t)| + \sum_{i \in I_{equal}(t)} \text{SIGN}(x_i(t) - u_i(t)) \right| \geq \alpha. \tag{43}$$

Additionally, (35) and (37) also imply that

$$\begin{aligned} \text{sign} \left(|I_{down}(t)| - |I_{up}(t)| + \sum_{i \in I_{equal}(t)} \text{SIGN}(x_i(t) - u_i(t)) \right) \\ = \text{sign} \left(|I_{down}(t)| - |I_{up}(t)| \right). \end{aligned}$$

By exploiting the definition of the median value, and considering (33) and (35) one derives the following implications

$$|I_{down}(t)| < |I_{up}(t)| \quad \Longleftrightarrow \quad c(t) < \inf \mathcal{M}(u(t)), \tag{44}$$

$$|I_{down}(t)| > |I_{up}(t)| \quad \Longleftrightarrow \quad c(t) > \sup \mathcal{M}(u(t)). \tag{45}$$

Relations (44) and (45) imply in turn that for $|I_{down}(t)| \neq |I_{up}(t)|$

$$\text{sign} \left(|I_{down}(t)| - |I_{up}(t)| \right) = \text{sign} \left(c(t) - m(u(t)) \right). \tag{46}$$

Therefore, under Assumption 3.2, by Proposition 3.3, and by eq.s (32), (43) and (46), it holds $\max_{\xi \in \widetilde{\mathcal{L}} V_2(t)} \xi \leq -\frac{\alpha}{n} + \Pi$. Thus, since $\frac{d}{dt}\left(V_2(t)\right) \in \widetilde{\mathcal{L}} V_2(t)$, unless $c(t) \in \mathcal{M}(u(t))$ (i.e., $V_2(t) \leq L_M$) it holds

$$\frac{d}{dt}\left(V_2(t)\right) \leq -\frac{\alpha}{n} + \Pi. \tag{47}$$

Thus, in Case 1, the finite time achievement of condition $c(t) \in \mathcal{M}(u(t))$ is guaranteed by analogous developments as those made in the proof of Theorem 4.1, with a finite transient time satisfying (21).

Case 2. This case may only happen in the event that more than one agent holds the same value of their corresponding reference signal, $u_i(t)$, and additionally this value corresponds to the median value. We now prove that if condition (36) holds then $c(t) \in \mathcal{M}(u(t))$. Define

$$k_{up}(t) = |I_{up}(t)| - n + 1, \quad k_{down}(t) = |I_{down}(t)|. \tag{48}$$

Thus

$$|I_{equal}(t)| = (k_{up}(t) - k_{down}(t)) - 1. \tag{49}$$

Relation (36) yields the two inequalities

$$|I_{up}(t)| - |I_{down}(t)| \leq |I_{equal}(t)| \quad \text{if} \quad |I_{up}(t)| > |I_{down}(t)| \tag{50}$$

$$|I_{down}(t)| - |I_{up}(t)| \leq |I_{equal}(t)| \quad \text{if} \quad |I_{up}(t)| < |I_{down}(t)| \tag{51}$$

By substituting (48), and (49) into (50) it yields

$$|I_{up}(t)| \leq \frac{n}{2} \tag{52}$$

Since we are investigating the case $|I_{up}(t)| > |I_{down}(t)|$, it derives from (52) and (48) that $k_{down}(t) < \frac{n}{2}$. Thus, if $|I_{up}(t)| > |I_{down}(t)|$ then the set $I_{equal}(t)$ contains the index of at least one agent whose state belongs to the median interval $\mathcal{M}(u(t))$ (or is equal to the median value $m(u(t))$ if n is odd). It follows that, since the network is at consensus, all agents in set $I_{equal}(t)$ have state value equal to the median value.

By applying similar considerations to (51) one can derive the same result which implies that $c(t) \in \mathcal{M}(u(t))$.

To evaluate the convergence time, let us write down the inequality

$$\begin{aligned} V_2(t) = V_2(T) + \int_T^t \frac{d}{dt}\left(V_2(t)\right) dt \leq V_2(T) + \int_T^t \left(-\frac{\alpha}{n} + \Pi \right) dt \\ = V_2(T) - \left(\frac{\alpha}{n} - \Pi \right)(t - T). \end{aligned}$$

By letting $V_2(t) \leq L_M(t)$, for $t = T_2$ it holds $V_2(T) - \left(\frac{\alpha}{n} - \Pi\right)(T_2 - T) \leq L_M(T_2)$, thus

$$T_2 \leq \frac{V_2(T) - L_M(T_2)}{\left(\frac{\alpha}{n} - \Pi\right)} + T = \frac{|c(T) - m(u(T))| - L_M(T_2)}{\frac{\alpha}{n} - \Pi} + T. \tag{53}$$

■

Next, we consider the case where agents join and leave the network according to Assumptions 3.6 and 3.7.

***Theorem** 4.3: Consider the network dynamics* (5) *along with time-varying graph $\mathcal{G}(t)$ satisfying Assumption 3.5. Assume that agents join and leave the network according to Assumption 3.6 and 3.7. Let the tuning parameters, under Assumption 3.2, satisfy $n_{max}\Pi < \alpha < \frac{2\lambda}{n_{max}}$ and let the consensus condition* (15) *be in force starting from time $t = T$. Under Assumption 3.4, there exists T_3 such that*

$$|c(t) - m(u(t))| \leq B, \quad \forall t \geq T_3, \tag{54}$$

where $B \leq \left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau$ *and*

$$T_3 \leq \left(\frac{\max\{|c(T) - m(u(T))| - B, 0\}}{\left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau - B}\right)\Delta\tau + T. \tag{55}$$

Proof: According to Assumption 3.6 there exists a minimum dwell time $\Delta\tau$ between two consecutive join or leave events. Consider two instants of time, t' where $|\mathcal{V}(t')| = n(t')$, and $t'' = t' + \Delta\tau$ where $|\mathcal{V}(t'')| = n(t'')$. Thus for $t \in (t', t'')$ there is at most one agent that either joins or leaves the network at the isolated instant of time $\hat{t} \in (t', t'')$. Thus, we can apply the results of Theorem 4.2 and disregard the isolated instant $\hat{t}$ in the Lebesgue integration of the Lyapunov function. In particular, by Assumption 3.7 (maximum number n_{max} of agents in the network), it holds $\frac{d}{dt}(V_2(t)) \leq -\frac{\alpha}{n_{max}} + \Pi$. By Assumption 3.6, $|n(t'') - n(t')| \leq 1$. Now, we compute the variation of $V_2(t)$ at time $\hat{t}$ when the join or leave event occurs. It holds $V_2(\hat{t}^+) - V_2(\hat{t}) = |c(\hat{t}^+) - m(u(\hat{t}^+))| - |c(\hat{t}) - m(u(\hat{t}))|$. Since $c(\hat{t}^+) = c(\hat{t})$, because agents join with state value equal to the average of their neighbors and the consensus condition is enforced for all $t \geq T$, we can write $V_2(\hat{t}^+) - V_2(\hat{t}) = |c(\hat{t}) - m(u(\hat{t}^+))| - |c(\hat{t}) - m(u(\hat{t}))|$. Furthermore, by Assumption 3.4, it holds $|m(u(\hat{t}^+)) - m(u(\hat{t}))| \leq B$. Thus $c(\hat{t}) - m(u(\hat{t}^+)) \leq c(\hat{t}) - m(u(\hat{t})) + B$ and

$$\begin{aligned} |V_2(\hat{t}^+) - V_2(\hat{t})| &\leq \big||c(\hat{t}) - m(u(\hat{t})) + B| - |c(\hat{t}) - m(u(\hat{t}))|\big| \\ &\leq \big||c(\hat{t}) - m(u(\hat{t}))| + B - |c(\hat{t}) - m(u(\hat{t}))|\big| \\ &= B \end{aligned} \tag{56}$$

We now integrate the Lyapunov function during an interval of length equal to the dwell time $\Delta\tau$ by taking into account (47) and (56) and disregarding the set of instants of time of measure zero in which graph $\mathcal{G}(t)$ changes, according to Assumption 3.5. It holds

$$\begin{aligned} V_2(t' + \Delta\tau) &= V_2(t') + \int_{t'}^{\hat{t}} \frac{d}{dt}(V_2(t))\,dt + V_2(\hat{t}^+) - V_2(\hat{t}) \\ &\quad + \int_{\hat{t}^+}^{t'+\Delta\tau} \frac{d}{dt}(V_2(t))\,dt \\ &\leq V_2(t') + \left(-\frac{\alpha}{n_{max}} + \Pi\right)(\hat{t} - t') + B \\ &\quad + \left(-\frac{\alpha}{n_{max}} + \Pi\right)(t' + \Delta\tau - \hat{t}^+). \end{aligned} \tag{57}$$

Since $\hat{t}^+ - \hat{t} = 0$, it holds $V_2(t' + \Delta\tau) \leq V_2(t') - \left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau + B$. If we consider time $t' = T$ as the initial instant of time in which consensus is enforced, then there exists $k \in \mathbb{N}^+$ such that

$$V_2(T + k\Delta\tau) \leq V_2(T) - k\left(\left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau - B\right). \tag{58}$$

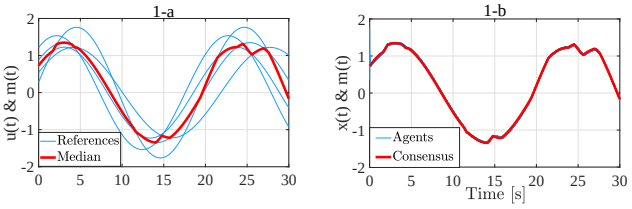


Fig. 1. (1-a) Reference signals $u_i(t)$ and their median value $m(u(t))$; (1-b) Time evolution of the network state $x(t)$.

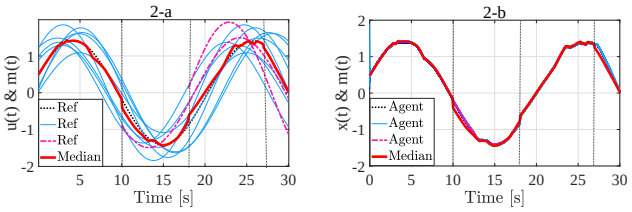


Fig. 2. (2-a) Reference signals $u_i(t)$ and their median value $m(u(t))$; (2-b) Time evolution of the network state $x(t)$.

Thus, if $B \leq \left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau$ then there exists k' and $T_3 \leq k'\Delta\tau + T$ such that by letting $V_2(T_3) \leq B$, it holds $V_2(T) - k'\left(\left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau - B\right) \leq B$, thus

$$k' \geq \frac{\max\{V_2(T) - B, 0\}}{\left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau - B} = \frac{\max\{|c(T) - m(u(T))| - B, 0\}}{\left(\frac{\alpha}{n_{max}} - \Pi\right)\Delta\tau - B}. \tag{59}$$

and $T_3 \leq k'\Delta\tau + T$. ■

According to Theorem 4.3 a procedure to tune the parameters of the proposed protocol to guarantee the achievement of dynamic consensus in finite time in the *open* scenario with bounded error is simply to choose a sufficiently large α so that $\alpha > n_{max}\Pi$, then choose a sufficiently large λ so that $\lambda > n_{max}\alpha$.

V. Simulation Results

In this section we present two numerical tests. In the first one, we consider a network of 5 agents connected by a random graph. We choose the tuning gains of the protocol according to Theorem 4.1 as $\lambda = 90$, $\alpha = 8$ and the initial states of the agents are choosen uniformly at random in the interval $[0, 2]$. We considered reference signals equal to $u_i(t) = a_i sin(2\pi f_i t)$ with coefficients chosen uniformly at random in the intervals $a_i \in [0, 2]$, $f_i \in [0, 0.05]$. In Figure 1-a, the reference signals and their median value are shown. In Figure 1-b it is shown the consensus value of the agents, which is shown to track, after a finite transient time (approximately 0.03 seconds, less than the analytical upper bound of $T_1 = 0.16$ seconds), the median value of the reference signals, starting at time $t = 0.6$ seconds.

In the second test, we consider the scenario of an open multi-agent system. The simulation starts with 9 agents at the consensus state on the median value $m(u(t))$ of their reference signals, then at $t = 10$ an agent joins the network with initial state according to (6) and $u_{10}(t) = 1.3sin(2\pi 0.048t + 0.9)$. Furthermore, at $t = 18$ sec a second agent joins the network, and finally at $t = 27$ sec an agent leaves the network. It can be seen in Figures 2-a and 2-b that when agents join or leave the network the median value of the

reference signals changes. In Figure 2-b, it is shown that the state of the agents remains at the consensus state and at bounded distance from $m(u(t))$ during the join and leave events.

VI. Conclusions

In this paper, we considered the dynamic consensus problem on median value in an open multi-agent system under a switching network topology. The considered protocol is shown to track with zero error in finite-time the median value of a set of time-varying reference signals with bounded variations. Furthermore, it is shown that when agents join and leave the network with an appropriate dwell time, the consensus value tracks with bounded error the median value. Future works will investigate discrete time versions of the proposed approach and its resilience with respect to faults or malicious agents.

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