Robust Containment Control in Multi-Agent Systems with Common Coordinate Frames and Bearing Angle Measurements

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***Abstract*— In this paper, the robust containment control problem in multi-agent systems (MAS) with multiple static leaders and with malicious agents is addressed. In our setting, we define as malicious all those agents which do not implement the local interaction protocol executed by the followers in the MAS. On the contrary, we assume malicious agents apply a control input of their own choice with the intent of jeopardizing the cooperation in order to bring the followers arbitrarily away from a containment area, i.e., an hypercube defined by the location of the leaders. For this setting, a distributed protocol, which is proven to be robust against malicious agents under certain topological conditions, is considered. It is assumed that the agents move in a d-dimensional hyperplane, have a common coordinate system, do not have access to absolute positions (GPS) and are able to measure bearing angles of their neighbors with respect to their own inertial reference frame. A theoretical characterization of the proposed algorithm is provided together with numerical results.**

I. Introduction

Motivation. Multi-agent systems (MAS) consist in large scale networks of dynamical systems that interact through a communication or sensing networks. Formal models of MAS have attracted huge interest due to their capability to model large networks of mobile robots or swarms, see for instance [1]–[3]. Indeed, MAS are envisioned to be exploited in futuristic scenarios where a large number of robots provides resiliency, flexibility and cost reduction as opposed to more capable and expensive robotic units. One of such futuristic applications consists in swarms of autonomous drones that thanks to eventual improvements of their intelligence enable a single pilot to control hundreds of them [4].

A major motivation to find strength in numbers is the supposed greater robustness of such systems with respect to failures of single agents or cyber-attacks toward single units. Unfortunately, most formal models of MAS used to represent and control such systems show that the behavior that emerges by the local interactions of agents through sensing or communication is extremely vulnerable to the malfunction or maliciousness of single agents, i.e., a single agent can disrupt the emergent behavior of a large scale network by choosing movements different from what would be prescribed by the local control protocols of the networks.

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In this paper, we propose and characterize a local interaction protocol that enables a MAS composed by a set of leaders and a set of followers to be robust against the existence of malicious agents, i.e., agents which have suffered a cyber-attack, have been hijacked or made unable to execute their prescribed local interaction protocol. We consider the containment control problem, i.e., the problem of designing a local interaction protocol such that the set of followers remains confined within an area or volume controlled by the leaders. We assume that the agents share a common coordinate system so that our proposed protocol can be execute on independent coordinate axes. Methods to achieve common coordinate frames exist, such as the one presented in [5], but require measurement of relative distance and bearing angles instead of just bearing angles.

Our main contribution can be summarized as follows.

- We consider a local interaction protocol based on a combination of sign functions of relative state differences and characterize its performance as containment control for a MAS.
- We consider malicious agents that interact with oblivious followers and propose conditions on the topology of the network which guarantee the robustness of the containment control.
- The agents are assumed to share a common coordinate system, do not have access to their global positions and are able to measure bearing angles with respect to their neighbors in their local inertial reference frame.
- The agents implement the proposed protocol independently on each movement axis, in this sense the results hold in a general d-dimensional space.

Literature review. Our considered control strategy is a discontinuous control action based on the sign function which has been investigated in several forms to solve consensus problems. Distributed control protocols which exploit the sign function can be found in [6]–[8]. Distributed control protocols based on the sign function have shown a degree of robustness with respect to noise and uncertain dynamics [9], [10], and regarding the consensus problem robustness to outliers or oblivious uncooperative agents [9], [11], [12].

Examples of containment control strategies of MAS can be found in [13], [14]. In [13] the authors study the containment control problem in the context of partial difference equations and Laplacian feedback, proposing a stop-and-go policy to move the followers while keeping the containment active. In [14] the authors consider instead a Lyapunov based approach applied to linear local interaction protocols under fixed and

switching network topologies.

In particular, our proposed robust interaction protocol takes inspiration from [9] where a combination of sign functions and tuning parameters was proven to solve the consensus on the median value problem, being the median value a statistic robust against outliers as opposed to the average. In [9] it was further shown under a rather restrictive set of assumption of the network topology, denoted as k-safe network, it is possible to make the achievement of the consensus state of a MAS robust to uncooperative agents. In this paper, as opposed to [9], we consider a different local interaction protocol where agents are either leaders or followers, all are oblivious to their absolute position and tasked to solve a containment control problem with presence of malicious agents.

In this paper we consider a scenario where the malicious agents are not identified and therefore can not be disconnected from the network. Other authors propose fault detection and identification algorithms, such as in [15], where non-cooperative agent can be identified and thus disconnected from the network.

It is also worth to mention that an other direction of research on the topic consists in approaches that aim to add robustness in the context of distributed optimization based on the so-called notion of r-robustness to deal with adversarial nodes, see for instance [16] and references therein. These generic protocols basically consider a large set of neighbors for each agent and in their local state updates discard most of the state values corresponding to far neighbors as opposed to close neighbors. If certain topological conditions are verified, these protocols tend to be robust with respect to faulty agents or uncooperative agents with specific behaviors such as moving away from the MAS.

Structure of the paper. In Section II preliminaries on network modeling and non-smooth analysis are given. In Section III we state the robust containment control problem and propose a local interaction protocol able to solve the problem. In Section IV a theoretical characterization of the convergence properties of the proposed protocol design is discussed. In Section V numerical simulations that corroborate the theoretical analysis are shown and finally in Section VI concluding remarks are given.

II. Preliminaries

A. Network Modeling

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes representing agents and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges representing their ability to sense bearing angles among them. Let $(i, j) \in \mathcal{E}$ be the edge joining the agents i and j. Let $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ be the set of neighbors of agent i.

A *path* in a graph $\mathcal{G}$ is a sequence of consecutive edges connecting two agents. Two paths are said to be *edge-independent* if they do not have any internal edge in common. A graph is said to be *connected* if there exists a path between any pair of nodes. We now introduce the definitions of k-*connected* and $(k, \mathcal{S})$-*connected* graph.

Definition 1 (k-*connected* graph)**.** Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with node set $\mathcal{V}$ and edge-set $\mathcal{E}$. We say that $\mathcal{G}$ is k-*connected* if for any couple of nodes in $\mathcal{V}$ there exists at least k edge-independent paths connecting them. ■

Note that a simple way to achieve a k-*connected* network is to form a rigid graph in k dimensions (a rigid planar graph when $k = 2$, see [17]).

Definition 2 ($(k, \mathcal{S})$-*connected* graph)**.** Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with node set $\mathcal{V}$, edge-set $\mathcal{E}$ and a node-subset $\mathcal{S} \subseteq \mathcal{V}$. We say that $\mathcal{G}$ is $(k, \mathcal{S})$-*connected* if for every couple of nodes $\{i, j\} \in \mathcal{S}$ there exists at least k edge-independent paths connecting i and j. ■

It should be noticed that in a $(k, \mathcal{S})$-*connected* graph $\mathcal{G}$, a path connecting any two vertexes i and j with $\{i, j\} \in \mathcal{S}$ can have edges with endpoints outside of the subset $\mathcal{S}$. Fig. (1) shows an example of a 2-*connected* graph with 5 nodes that is also $(3, \mathcal{S})$-*connected* with respect to the subset of nodes $\mathcal{S} = \{1, 2, 3\}$.

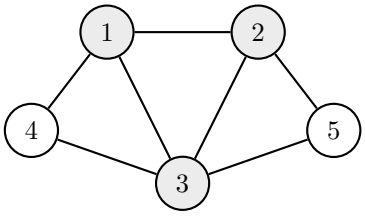


Fig. 1. Example of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ both 2-*connected* and $(3, \mathcal{S})$-*connected* with respect to the subset of gray nodes $\mathcal{S} = \{1, 2, 3\}$.

B. Non-smooth analysis

In this section, we review some fundamental results concerning non-smooth analysis that will be employed hereafter in the paper to investigate the convergence properties of the considered distributed protocol. The reader is referred to [18]–[22] and references therein for a comprehensive overview of the topic. Consider the (possibly discontinuous) dynamical system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad (1)$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, is defined almost everywhere, i.e., it is defined for every $x \in \mathbb{R}^n \setminus W$, where W is a subset of $\mathbb{R}^n$ of measure zero. Furthermore, $f(x)$ is measurable in an open region $Q \subset \mathbb{R}^n$ and for all compact sets $D \subset Q$ there exists a constant A_D such that $\|f(x)\| \leq A_D$ almost everywhere in D.

If the differential equation (1) has discontinuous right-hand side, following [22] we understand the corresponding solution in the so-called *Filippov sense* as the solution of an appropriate differential inclusion, as explained in the next definition.

Definition 3 (Filippov Solution)**.** A vector function $x(\cdot)$ is called solution of (1) on a time interval $[t_0, t_i]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_i]$ and for almost all $t \in [t_0, t_i]$

$$\dot{x} \in K[f](x) \qquad (2)$$

where $K[f](x) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined as

$$K[f](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu\{H\}=0} \overline{\text{co}}\{f(B(x,\delta) \setminus H)\} \tag{3}$$

where $\bigcap_{\mu\{H\}=0}$ denotes the intersection over all sets H of Lebesgue measure zero, $B(x,\delta)$ denotes the ball of radius δ centered at x, $\overline{\text{co}}$ the convex closure and $2^{\mathbb{R}^n}$ the set of subsets of $\mathbb{R}^n$.

Briefly, the idea of the Filippov's solution is that the tangent vector to a solution, where it exists, must lie in the convex closure of the values of the vector field in progressively smaller neighborhoods around the solution point. A very important aspect of this definition is given by the possibility of discarding sets of measure zero. Indeed, this technical detail allows solutions to be defined even at points where the vector field itself is not defined.

If $f(x)$ is measurable and locally bounded then the set-valued map $K[f](x)$ is upper semicontinuous, compact, convex valued and locally bounded so that the differential inclusion (2) possesses a Filippov solution for each initial condition x_0. The reader is referred to [20] for a comprehensive tutorial on the different alternative solution notions for discontinuous dynamical systems, and to [21] for a comprehensive theory of generalized gradients and their applications.

Let us now recall the definition of the *Clarke's Generalized Gradient*.

Definition 4 (Clarke's Generalized Gradient [21])**.** Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient $\partial V(x)$ is defined as

$$\partial V(x) \triangleq co\left\{\lim_{i\to\infty} \nabla V(x_i) | x_i \to x, x_i \notin \Omega_V \cup N\right\},$$

where ∇V denotes the conventional gradient, $x_i \in \mathbb{R}^n$ represents a point of an infinite succession which converges to $x \in \mathbb{R}^n$ as i grows to infinity, Ω_V is a set of Lebesgue measure zero which contains all points where $\nabla V(x)$ does not exist, and N is an arbitrary set of measure zero. ■

We now review the chain rule which allows to differentiate Lipschitz regular functions along the Filippov's solution trajectories.

Theorem 1 (Chain Rule [19])**.** *Let $x(\cdot)$ be a Filippov solution to* (1) *and $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and, in addition, regular function. Then $V(x)$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) \tag{4}$$

where the generalized time derivative $\dot{\tilde{V}}(x)$ is defined as

$$\dot{\tilde{V}}(x) := \bigcap_{\xi\in\partial V(x(t))} \xi^T K[f](x) \tag{5}$$

Let us now recall a revised version of the Generalized Lyapunov theorem given in [18] based on the results given in [19]. This will prove useful to establish finite-time stability results for dynamical systems described by differential equations with discontinuous right-hand side.

Theorem 2 (Stability Theorem)**.** *Let $x(\cdot)$ be a Filippov solution to* (1) *and $V(x) : \mathbb{R}^n \to \mathbb{R}$, be a time independent regular function such that $V(x) = 0\ \forall x \in \mathcal{C}$ and $V(x) > 0\ \ \forall x \notin \mathcal{C}$, with $\mathcal{C} \in \mathbb{R}^n$ a compact set. Let $x : \mathbb{R} \to \mathbb{R}^n$ and $V(x(t))$ be absolutely continuous on $[t_0, \infty)$ with $\dot{\tilde{V}}(x) \leq -\mu < 0$ with $\mu > 0$ for $\forall x \notin \mathcal{C}$. Then, $V(x(t))$ converges to 0 in finite time and $x(t)$ reaches the compact set $\mathcal{C}$ in finite time as well.*

At this point, let us now define the discontinuous "sign" function, the discontinuous and set-valued "SIGN" function and the discontinuous vector "**sign**" function as follows

$$\text{sign}(y) = \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ -1, & \text{if } y < 1, \end{cases} \qquad y \in \mathbb{R}, \tag{6}$$

and

$$\text{SIGN}\,(y) \in \begin{cases} 1 & \text{if} \quad y > 0, \\ [-1,1] & \text{if} \quad y = 0, \\ -1 & \text{if} \quad y < 0. \end{cases} \qquad y \in \mathbb{R}, \tag{7}$$

and

$$\mathbf{sign}(y) = \Big[\text{sign}(y_1), \cdots, \text{sign}(y_n)\Big]^T, \qquad y \in \mathbb{R}^n. \tag{8}$$

III. Problem Setting

Let us consider a multi-agent system where three different sets of agents are considered, namely followers, leaders and attackers. More specifically, the set of "followers", denoted as $\mathcal{V}_f$, represents the agents that execute a distributed protocol based on the sensing of the bearing angles of their neighbors with respect to their own inertial reference frame. Furthermore, the set of "leaders", denoted as $\mathcal{V}_\ell$, represents the agents which define the containment area that the followers are expected to reach and not escape from. Finally, the set of "attackers", denoted as $\mathcal{V}_a$ represents malicious agents tasked with preventing the followers to reach and/or remain within the containment area defined by the position of the leaders. Clearly the node set $\mathcal{V}$ of the network $\mathcal{G}$ is such that $\mathcal{V} = \mathcal{V}_f \bigcup \mathcal{V}_\ell \bigcup \mathcal{V}_a$. In the rest of the paper, the following additional set $\mathcal{V}_{f\ell} = \mathcal{V}_f \bigcup \mathcal{V}_\ell$ will be used to denote the set of cooperative followers and leaders together.

In particular, followers, leaders and malicious agents have respectively the following dynamics

$$\dot{x}_i = u_i(t) \qquad \forall\, i \in \mathcal{V}_f \tag{9}$$

$$\dot{x}_i = 0_d \qquad \forall\, i \in \mathcal{V}_\ell \tag{10}$$

$$\dot{x}_i = a_i(t) \qquad \forall\, i \in \mathcal{V}_a \tag{11}$$

where $u_i(t) \in \mathbb{R}^d$ is the local control input of the follower x_i to be later specified, 0_d is the zero vector of dimension d, and $a_i(t) \in \mathbb{R}^d$ is the dynamic of the malicious agent x_i.

In the sequel, for the sake of clarity and with no lack of generality we will collect the agents by sorting and labelling them according to their kind. As a result, we will denote the stacked vector of the followers state with

$\mathbf{x}^f = \left[x_1^T, \ldots, x_n^T\right]^T$ where $n = |\mathcal{V}_f|$, the stacked vector of the leaders state with $\mathbf{x}^\ell = \left[x_{n+1}^T, \ldots, x_{n+s}^T\right]^T$ where $s = |\mathcal{V}_\ell|$, and the stacked vector of the attackers state with $\mathbf{x}^a = \left[x_{n+s+1}^T, \ldots, x_{n+s+r}^T\right]^T$ where $r = |\mathcal{V}_a|$. In addition, for the sake of the analysis we will denote the stacked vector of followers and leaders state as $\mathbf{x} = \left[\mathbf{x}^{fT}\ \mathbf{x}^{\ell T}\right]^T$. Furthermore, we will denote the stacked vector of the per-agent p-th components as $\mathbf{x}_p$.

We are now ready to state our problem formulation, that is our goal is to design a cooperative protocol $u_i(t)$, $i \in \mathcal{V}_f$ for the followers so that they can reach in finite time a containment region defined according to the position of the leaders $\mathcal{V}_\ell$.

Problem 1. Let us consider a containment region $\mathcal{C}(\mathbf{x}_\ell(0))$ defined according to the location of the leaders. Our goal is to design a distributed cooperative control law $u_i(t)$, $i \in \mathcal{V}_f$ for the followers so that for any initial condition $\mathbf{x}^f(0)$, $\mathbf{x}^\ell(0)$, $\mathbf{x}^a(0)$, the following holds true

$$x_i(t) \in \mathcal{C}\left(\mathbf{x}_\ell(0)\right),\ i \in \mathcal{V}_f,\ \forall\, t \geq T \tag{12}$$

with T the finite convergence time.

We now provide a distributed cooperative control law $u_i(t)$ that will be proven later to solve Problem 1 under proper topological conditions. In particular, we assume that the following control law $u_i(t)$ is implemented by each follower $i \in \mathcal{V}_f$

$$u_i(t) = -\sum_{j \in N_i} \mathbf{sign}\Big(x_i(t) - x_j(t)\Big) \quad \forall i \in \mathcal{V}_f \tag{13}$$

In the next section, we characterize: i) topological conditions under which protocol (13) can solve Problem 1; ii) the containment region $\mathcal{C}(\mathbf{x}_\ell(0))$ defined according to the location of the leaders, and iii) the (finite) time required to the followers to reach such a region.

IV. Theoretical Analysis

In order to characterize the properties of the considered protocol, collected in Theorem 3, let us first introduce the sets $\mathcal{I}_p^M(t)$ and $\mathcal{I}_p^m(t)$ collecting the followers and the leaders with maximum and minimum value on the p-th component defined as

$$\mathcal{I}_p^M(t) = \left\{ i \in \mathcal{V}_{f\ell} : x_{i,p} = \max_{j \in \mathcal{V}_{f\ell}} x_{j,p}(t) \right\} \tag{14}$$

$$\mathcal{I}_p^m(t) = \left\{ i \in \mathcal{V}_{f\ell} : x_{i,p} = \min_{j \in \mathcal{V}_{f\ell}} x_{j,p}(t) \right\} \tag{15}$$

and the variables $\overline{x}_p$ and $\underline{x}_p$ representing the maximum and minimum value among the leaders along the p-th component

$$\overline{x}_p(t) = \max_{i \in \mathcal{V}_\ell} x_{i,p}(0) \tag{16}$$

$$\underline{x}_p(t) = \min_{i \in \mathcal{V}_\ell} x_{i,p}(0) \tag{17}$$

We point out that since the dynamics of the leaders in eq. (10) are null, the variables $\overline{x}_p(t)$ and $\underline{x}_p(t)$ assume the same value for all $t \geq 0$.

We are now ready to state our result concerning the finite time convergence of the algorithm introduced in eq. (13).

Theorem 3. *Consider a multi-agent system described by an undirected* $(k, \mathcal{V}_{fl})$-connected *network* $\mathcal{G}$ *composed of* $|\mathcal{V}_f|$ *followers,* $|\mathcal{V}_\ell|$ *leaders, and* $|\mathcal{V}_a|$ *malicious agents with dynamics eqs.* (9) *to* (11)*. Let us assume that the followers run the proposed protocol* (13) *and let* α *be the maximum degree of the malicious agents* $\mathcal{V}_a$*. Then, if*

$$k = 2\,\alpha\,|\mathcal{V}_a| + \beta, \quad \beta \geq 1 \tag{18}$$

there exists a finite time $T > 0$ *such that*

$$x_i(t) \in \mathcal{C}, \quad \forall i \in \mathcal{V}_f, \quad \forall t \geq T \tag{19}$$

where the hypercube $\mathcal{C}$ *is defined as*

$$\mathcal{C} = \left\{ y \in \mathbb{R}^d\ :\ y_{i,p} \in \left[\underline{x}_p, \overline{x}_p\right],\ p \in \{1, \ldots, d\} \right\} \tag{20}$$

and the convergence time T *is bounded by* $\overline{T}$ *defined as*

$$\overline{T} = \max_{p \in \{1,\ldots,d\}} \left\{ \frac{|\mathcal{V}_f|}{\beta} \max\left\{ V_p^M(\mathbf{x}_p(0)),\, V_p^m(\mathbf{x}_p(0)) \right\} \right\} \tag{21}$$

in which the terms $V_p^M(\mathbf{x}_p(0))$ *and* $V_p^m(\mathbf{x}_p(0))$ *are*

$$\begin{aligned} V_p^M(\mathbf{x}_p(0)) &= \sum_{i \in \mathcal{I}_p^M(0)} \left| x_{i,p}(0) - \overline{x}_p(0) \right| \\ V_p^m(\mathbf{x}_p(0)) &= \sum_{i \in \mathcal{I}_p^m(0)} \left| x_{i,p}(0) - \underline{x}_p(0) \right| \end{aligned} \tag{22}$$

Proof. In order to prove this result we analyze the behaviour of the system along its p-th component. To this end, let us introduce the two following nonsmooth Lyapunov functions $V_p^M(\mathbf{x}_p(t)) : \mathbb{R}^{n+s} \to \mathbb{R}_+$ and $V_p^m(\mathbf{x}_p(t)) : \mathbb{R}^{n+s} \to \mathbb{R}_+$ defined as

$$V_p^M(\mathbf{x}_p(t)) = \frac{1}{|\mathcal{I}_p^M(t)|} \sum_{i \in \mathcal{I}_p^M(t)} V_{i,p}^M(\mathbf{x}_p(t)) \tag{23}$$

$$V_p^m(\mathbf{x}_p(t)) = \frac{1}{|\mathcal{I}_p^m(t)|} \sum_{i \in \mathcal{I}_p^m(t)} V_{i,p}^m(\mathbf{x}_p(t)) \tag{24}$$

in which the terms $V_{i,p}^M(\mathbf{x}_p(t))$ and $V_{i,p}^m(\mathbf{x}_p(t))$ are

$$V_{i,p}^M(\mathbf{x}_p(t)) = \left| x_{i,p}(t) - \overline{x}_p(t) \right| \tag{25}$$

$$V_{i,p}^m(\mathbf{x}_p(t)) = \left| x_{i,p}(t) - \underline{x}_p(t) \right| \tag{26}$$

Moreover, let us introduce the p-component of the hypercube $\mathcal{C}$ as

$$\mathcal{C}_p = \left\{ y \in \mathbb{R} : y \in \left[\overline{x}_p, \underline{x}_p\right] \right\} \tag{27}$$

In virtue of the previous definitions, the sum of the two Lyapunov functions is positive as long as there exists a follower $x_{i,p}(t) \in \mathcal{I}_p^M(t) : x_{i,p}(t) \notin \mathcal{C}_p$ or a follower $x_{i,p}(t) \in \mathcal{I}_p^m(t) : x_{i,p}(t) \notin \mathcal{C}_p$.

In what will follow, dependence on time t and state $\mathbf{x}$ will be used only when introducing new concepts or variables and omitted otherwise for the sake of clarity.

Let us now first analyze the term V_p^M. In particular, consider the case when $V_p^M > 0$, i.e. there exists at least a follower $x_{i,p} \in \mathcal{I}_p^M$ such that $x_{i,p} > \overline{x}_p$. Since V_p^M is a

Lipschitz regular function we can resort to the chain rule given in Theorem 1 to compute the generalized time derivative as

$$\dot{\tilde{V}}_p^M(\mathbf{x}_p(t)) = \bigcap_{\xi \in \partial V(\mathbf{x}_p(t))} \xi^T K\left[f\right](\mathbf{x}_p(t)) \tag{28}$$

where the generalized gradient $\partial V_p^M(\mathbf{x}_p)$ assumes the form

$$\partial V_p^M(\mathbf{x}_p) \subseteq \sum_{i \in \mathcal{I}_p^M} \partial V_{i,p}^M(\mathbf{x}_p) \tag{29}$$

in which the single element $\partial V_{i_p}^M(\mathbf{x}_p)$ is defined as

$$\partial V_{i,p}^M(\mathbf{x}_p) \subseteq \left[0, \ldots, \frac{\text{SIGN}\left(x_{i,p} - \overline{x}_p\right)}{\left|\mathcal{I}_p^M\right|}, \ldots, 0\right]^T \tag{30}$$

The overall form for $\partial V_p^M(\mathbf{x}_p)$ is thus

$$\partial V_p^M(\mathbf{x}_p) \subseteq \left[\frac{\text{SIGN}\left(x_{1,p} - \overline{x}_p\right)}{\left|\mathcal{I}_p^M\right|}, \ldots, \frac{\text{SIGN}\left(x_{n+s,p} - \overline{x}_p\right)}{\left|\mathcal{I}_p^M\right|}\right]^T \tag{31}$$

Therefore, from the previous derivations it follows that a generic $\xi_p \in \partial V_p^M(\mathbf{x}_p)$, which can be expressed as $\xi_p = [\xi_{p,1}, \ldots, \xi_{p,n+s}]^T$, is such that

$$\xi_{p,i} \in \frac{\text{SIGN}\left(x_{i,p} - \overline{x}_p\right)}{\left|\mathcal{I}_p^M\right|} \tag{32}$$

Similarly, the set valued map $K[f](\mathbf{x}_p)$ is

$$K[f](\mathbf{x}_p) \subseteq [K[f_1](\mathbf{x}_p), \ldots, K[f_{n+s}](\mathbf{x}_p)]^T \tag{33}$$

in which the single term $K[f_i](\mathbf{x}_p)$ is defined as follows

$$\begin{aligned} K[f_i](\mathbf{x}_p) &\subseteq K\left[-\sum_{j \in N_i} \text{sign}(x_{i,p} - x_{j,p})\right] && \forall i \in \mathcal{V}_f \\ &\subseteq -\sum_{j \in N_i} \text{SIGN}(x_{i,p} - x_{j,p}) \\ K[f_i](\mathbf{x}_p) &\subseteq \{0\} && \forall i \in \mathcal{V}_\ell \end{aligned} \tag{34}$$

in virtue of the agents dynamics in eqs. (9) and (10) and the definition of $\overline{x}_p$ in eq. (16).

The generalized time derivative $\dot{\tilde{V}}_p^M(\mathbf{x}_p)$ introduced in eq. (28) can then be restated as

$$\dot{\tilde{V}}_p^M(\mathbf{x}_p) \subseteq \bigcap_{\xi_p \in \partial V_p^M} \xi_p^T \; K\left[f\right](\mathbf{x}_p) \tag{35}$$

At this point, exploiting eqs. (32) and (34) we get

$$\begin{aligned} \xi_p^T \; K\left[f\right](\mathbf{x}_p) &\subseteq \sum_{i \in \mathcal{I}_p^M} \xi_{p,i} \; K[f_i](\mathbf{x}_p) \\ &\subseteq \sum_{i \in \mathcal{I}_p^M} \left(\frac{\text{SIGN}\left(x_{i,p} - \overline{x}_p\right)}{\left|\mathcal{I}_p^M\right|} \left(-\sum_{j \in N_i} \text{SIGN}(x_{i,p} - x_{j,p}) \right) \right) \end{aligned} \tag{36}$$

We now differentiate eq. (36) with respect to the contributions of the neighbors N_i considering the three kinds of agents that make up the network, i.e. the followers $\mathcal{V}_f$, the leaders $\mathcal{V}_\ell$ and the malicious agents $\mathcal{V}_a$

$$\begin{aligned} &\sum_{i \in \mathcal{I}_p^M} \xi_{p,i} \; K[f_i](\mathbf{x}_p) \subseteq \\ &\frac{1}{\left|\mathcal{I}_p^M\right|} \sum_{i \in \mathcal{I}_p^M} \left(\text{SIGN}\left(x_{i,p} - \overline{x}_p\right) \left(-\sum_{j \in N_i \cap \mathcal{V}_f} \text{SIGN}(x_{i,p} - x_{j,p}) \right. \right. \\ &\left. \left. - \sum_{j \in N_i \cap \mathcal{V}_\ell} \text{SIGN}(x_{i,p} - x_{j,p}) - \sum_{j \in N_i \cap \mathcal{V}_a} \text{SIGN}(x_{i,p} - x_{j,p}) \right) \right) \end{aligned} \tag{37}$$

In addition, each one of these contributions can be split in two groups that represent the neighbors that are at maximum distance as the agent i and the ones that are not, that is

$$\begin{aligned} &\sum_{i \in \mathcal{I}_p^M} \xi_{p,i} \; K[f_i](\mathbf{x}_p) \subseteq \\ &\frac{1}{\left|\mathcal{I}_p^M\right|} \sum_{i \in \mathcal{I}_p^M} \left(\text{SIGN}\left(x_{i,p} - \overline{x}_p\right) \left(-\sum_{\substack{j \in N_i \cap \mathcal{V}_f \\ j \notin \mathcal{I}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) \right. \right. \\ &\left. \left. - \sum_{\substack{j \in N_i \cap \mathcal{V}_\ell \\ j \notin \mathcal{I}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) - \sum_{\substack{j \in N_i \cap \mathcal{V}_a \\ j \notin \overline{\mathcal{I}}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) \right) \right) \\ &+ \frac{1}{\left|\mathcal{I}_p^M\right|} \sum_{i \in \mathcal{I}_p^M} \left(\text{SIGN}\left(x_{i,p} - \overline{x}_p\right) \left(-\sum_{\substack{j \in N_i \cap \mathcal{V}_f \\ j \in \mathcal{I}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) \right. \right. \\ &\left. \left. - \sum_{\substack{j \in N_i \cap \mathcal{V}_\ell \\ j \in \mathcal{I}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) - \sum_{\substack{j \in N_i \cap \mathcal{V}_a \\ j \in \overline{\mathcal{I}}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) \right) \right) \end{aligned} \tag{38}$$

where the set $\overline{\mathcal{I}}_p^M$ collects the malicious agents with the same state of the agents belonging to the set $\mathcal{I}_p^M$ as

$$\overline{\mathcal{I}}_p^M = \left\{ h \in \mathcal{V}_a : x_{h,p} = x_{i,p},\, i \in \mathcal{I}_p^M \right\} \tag{39}$$

At this point we notice that eq. (38) can be further developed considering the following statements. First of all, since the graph $\mathcal{G}$ is undirected, the followers $x_{i,p} \in \mathcal{I}_p^M$ produce null contributions among themselves, that is

$$\sum_{i \in \mathcal{I}_p^M} \left(\text{SIGN}\left(x_{i,p} - \overline{x}_p\right) \left(\sum_{\substack{j \in N_i \cap \mathcal{V}_f \\ j \in \mathcal{I}_p^M}} \text{SIGN}(x_{i,p} - x_{j,p}) \right) \right) = 0 \tag{40}$$

Second, since the leaders $\mathcal{V}_\ell$ are static and we are considering $V_p^M(\mathbf{x}_p) > 0$ then the set $\mathcal{I}_p^M$ must be composed only of followers $\mathcal{V}_f$, i.e. $\mathcal{I}_k^M \cap \mathcal{V}_\ell = \emptyset$. Moreover, as a consequence of the definition of the set $\mathcal{I}_p^M$ in eq. (14) and the definition of

the variable $\overline{x}_p$ in eq. (16) we can clearly define the values of the following terms

$$\text{SIGN}(x_{i,p}-x_{j,p})=1 \quad \forall i \in \mathcal{I}_p^M, \forall j \in \left(N_i \cap \mathcal{V}_{f\ell} \setminus \mathcal{I}_p^M\right) \tag{41}$$

$$\text{SIGN}\left(x_{i,p}-\overline{x}_p\right)=1 \quad \forall i \in \mathcal{I}_p^M \tag{42}$$

Third, considering the worst case scenario, the malicious neighbors $x_{j,p} \notin \overline{\mathcal{I}}_p^M$ interactions can be bounded as

$$\sum_{i\in\mathcal{I}_p^M}\left(\sum_{\substack{j\in N_i\cap\mathcal{V}_a\\ j\notin\overline{\mathcal{I}}_p^M}}\text{SIGN}(x_{i,p}-x_{j,p})\right)\subseteq -\alpha\left|\mathcal{V}_a\setminus\overline{\mathcal{I}}_p^M\right|[-1,\,1] \leq \alpha\left|\mathcal{V}_a\setminus\overline{\mathcal{I}}_p^M\right| \tag{43}$$

and similarly, the malicious agents $x_{j,p} \in \overline{\mathcal{I}}_p^M$ interactions result in

$$\sum_{i\in\mathcal{I}_p^M}\left(\sum_{\substack{j\in N_i\cap\mathcal{V}_a\\ j\in\overline{\mathcal{I}}_p^M}}\text{SIGN}(x_{i,p}-x_{j,p})\right)\subseteq -\alpha\left|\mathcal{V}_a\cap\overline{\mathcal{I}}_p^M\right|[-1,\,1] \leq \alpha\left|\mathcal{V}_a\cap\overline{\mathcal{I}}_p^M\right| \tag{44}$$

Lastly, assume that there are q edges in the subgraph among the followers and the leaders connecting the agents $x_{i,p} \in \mathcal{I}_p^M$ and the agents $x_{j,p} \notin \mathcal{I}_p^M$. Then, eq. (38) can be restated as

$$\begin{aligned}\dot{\tilde{V}}_p^M(\mathbf{x}_p) &\leq \frac{1}{\left|\mathcal{I}_p^M\right|}\left(-q+\alpha\left|\mathcal{V}_a\setminus\overline{\mathcal{I}}_p^M\right|+\alpha\left|\mathcal{V}_a\cap\overline{\mathcal{I}}_p^M\right|\right)\\ &\leq \frac{1}{|\mathcal{V}_f|}\left(-q+\alpha\,|\mathcal{V}_a|\right)\end{aligned} \tag{45}$$

where α is an upper bound on the degree of the malicious agents $\mathcal{V}_a$. From eq. (45) it follows that if $q > \alpha\,|\mathcal{V}_a|$ the generalized time derivative is negative definite. At this point, by recalling that the network is $(k, \mathcal{V}_{f\ell})$-*connected*, it follows that it is sufficient to choose k such that

$$k = 2\,\alpha\,|\mathcal{V}_a| + \beta, \quad \beta \geq 1 \tag{46}$$

in order to satisfy the condition $q > \alpha\,|\mathcal{V}_a|$.

At this point, considering $\mu = q - \alpha\,|\mathcal{V}_a| = \beta$ and resorting to Theorem 2, we can conclude that the followers $x_{i,p} \in \mathcal{I}_p^M : x_{i,p} \notin \mathcal{C}_p$ move towards the containment region defined by the leaders along the p-dimension with a speed that is upper bounded by eq. (45) and that they eventually reach $\mathcal{C}_p$ in finite time.

Now, regarding the term V_p^m a similar analysis can be performed replacing the upper limit $\overline{x}_p$ and the sets $\mathcal{I}_p^M$, $\overline{\mathcal{I}}_p^M$ with the equivalent lower limit $\underline{x}_p$ and sets $\mathcal{I}_p^m$, $\overline{\mathcal{I}}_p^m$ where $\overline{\mathcal{I}}_p^m$ is defined analogously to the set $\overline{\mathcal{I}}_p^M$ as $\overline{\mathcal{I}}_p^m = \left\{h \in \mathcal{V}_a : x_{h,p} = x_{i,p},\, i \in \mathcal{I}_p^m\right\}$. Doing so allows us to obtain the following bound on $\dot{\tilde{V}}_p^m(\mathbf{x}_p)$

$$\dot{\tilde{V}}_p^m(\mathbf{x}_p) \leq -\frac{1}{|\mathcal{V}_f|}\beta \tag{47}$$

thus proving that in virtue of Theorem 2, the followers $x_{i,p} \in \mathcal{I}_p^m$ reach the lower limit $\underline{x}_p$ of the containment region $\mathcal{C}_p$ with a speed that is upper bounded by eq. (47).

In order to establish the two convergence time let us compute the following inequality:

$$\begin{aligned}V_p^h(\mathbf{x}_p(t)) &= V_p^h(\mathbf{x}_p(0)) + \int_0^t \dot{\tilde{V}}_p^h(x(\tau))\,\mathrm{d}\tau\\ &\leq V_p^h(\mathbf{x}(0)) - \int_0^t \frac{\beta}{|\mathcal{V}_f|}\,\mathrm{d}\tau\\ &= V_p^h(\mathbf{x}_p(0)) - \frac{\beta}{|\mathcal{V}_f|}\,t.\end{aligned} \tag{48}$$

where $h \in \{m, M\}$. An upper bound on the convergence time T is then

$$T_p = \frac{|\mathcal{V}_f|}{\beta}\max\left\{V_p^M(\mathbf{x}_p(0)),\, V_p^m(\mathbf{x}_p(0))\right\} \tag{49}$$

At this point, noticing that the evolution of the multi-agent system is decoupled along its d-components we can conclude that the followers converge to the containment area $\mathcal{C}$ defined by the location of the leaders with convergence time $T \leq \overline{T}$ where $\overline{T}$ is defined as

$$\overline{T} = \max_{p\,\in\{1,\ldots,d\}} T_p \tag{50}$$

the result thus follows. □

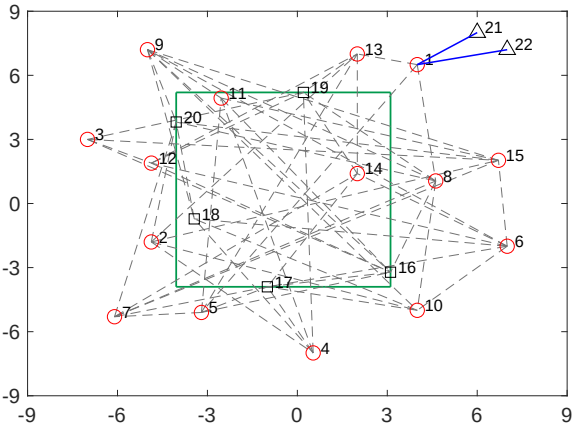
Note that, a sufficient condition to ensure negative definiteness of the generalized Lyapunov derivative is that the (sub)graph $\mathcal{G}_{f\ell} = \{\mathcal{V}_{f\ell}, \mathcal{E}_{f\ell}\}$ is *q-connected* with $q > \alpha|\mathcal{V}_a|$. However, in our setting, where agents have no knowledge about the role of others agents in the network, this can be safely implemented by assuming that the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is $(k, \mathcal{V}_{fl})$-*connected* with $k = 2\,\alpha\,|\mathcal{V}_a| + \beta$ where $\beta \geq 1$.

V. Numerical simulations

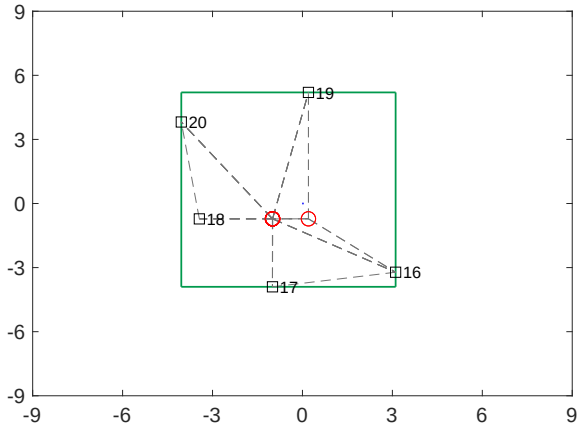
In this section, we provide numerical results to corroborate the theoretical findings. Two scenarios are considered: in the first setting, we show that Problem 1 can be effectively solved by following the conditions given in Theorem 3, while in the second setting we illustrate that malicious agents may prevent some followers to enter the containment area described by the position of the leaders when the conditions given in Theorem 3 do not hold.

Simulations are carried out in a 2-dimensional setting by considering a multi-agent system composed by $|\mathcal{V}_f| = 15$ followers, $|\mathcal{V}_\ell| = 5$ leaders and $|\mathcal{V}_a| = 2$ attackers with $\alpha = 1$.

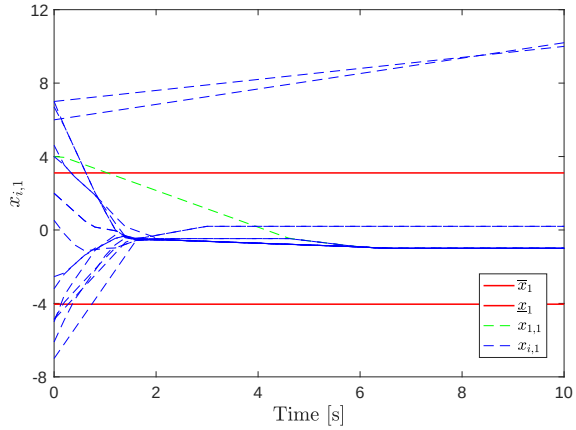
Fig. (2) describes the first simulation scenario where the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is $(k, \mathcal{V}_{f\ell})$-*connected* with $k = 5$, thus satisfying the requirements of Theorem 3. In particular, Fig. (2a) depicts the initial configuration of the agents, where followers, leaders and malicious agents are denoted with a circle, a square and a triangle, respectively, while Fig. (2b) describes the final configuration obtained by running protocol (13), where malicious agents are not shown as they moved far away from the containment region. Note that, in both Figs. (2a), and (2b), the containment region $\mathcal{C}(\mathbf{x}_\ell(0))$ (rectangle) defined by the location of the leaders is depicted in



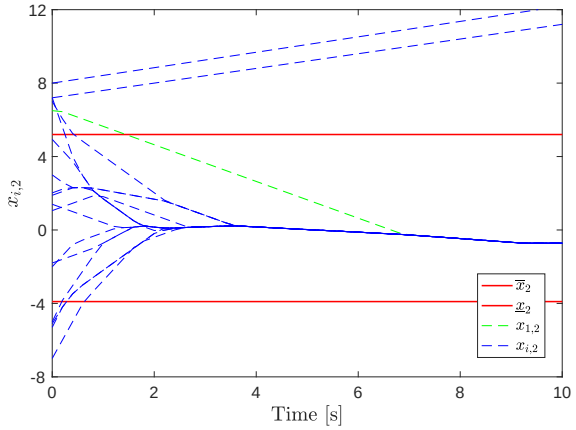
(a) Position of the team of agents at initial time $t = 0$ s.



(b) Position of the team of agents at final time $t = 10$ s.



(c) Evolution of the first component $x_{i,1}$ of the agents with respect to the limits $\overline{x}_1$ and $\underline{x}_1$ of the containment section $\mathcal{C}_1$.



(d) Evolution of the second component $x_{i,2}$ of the agents with respect to the limits $\overline{x}_2$ and $\underline{x}_2$ of the containment section $\mathcal{C}_2$.

Fig. 2. A multi-agent system composed of 15 followers, 5 leaders, 2 malicious agents with $\alpha = 1$ and graph $(5, \mathcal{V}_{f\ell})$-*connected*.

green, the edge-connectivity between any pair of cooperative agents is described with a gray dashed-line, while the edge-connectivity towards malicious agents is described by a blue solid-line. Figs. (2c) and (2d) shows the agents evolution along the x-and-y axes, respectively. It can be noticed that since the conditions of Theorem 3 hold, all followers are capable of reaching the containment area defined by the leaders position, despite the interactions with the malicious agents.

Fig. (3) describes the second simulation scenario where the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is $(k, \mathcal{V}_{f\ell})$-*connected* with $k = 4$, thus not satisfying the requirements of Theorem 3. Figs. (3a) and (3b) describes the initial configuration and final configuration, respectively. Figs. (3c) and (3d) shows the agents evolution along the x-and-y axes, respectively. It can be noticed that since the conditions of Theorem 3 are not satisfied, not all of the followers are capable of reaching the containment area defined by the leaders position due to the interactions with the malicious agents. In particular, agent 1 is now connected to two cooperative agents and two malicious agents that are moving towards the top right angle of the plane. As a result, as soon as agents 8 and 13 enter the containment region, its control action becomes identically equal to zero along both axis for all time, thus preventing it from moving.

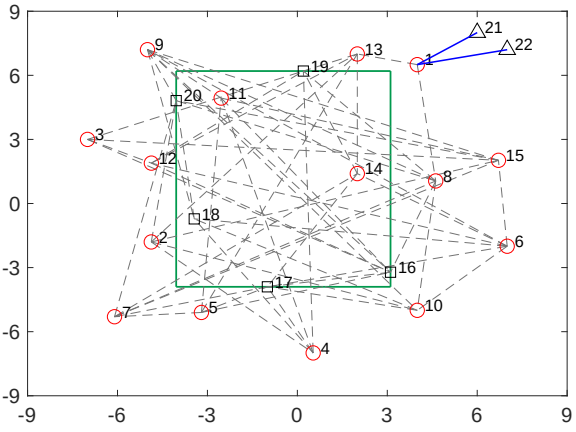
We reiterate that in our setting agents are not aware of the role of others agents in the network.

VI. Conclusions

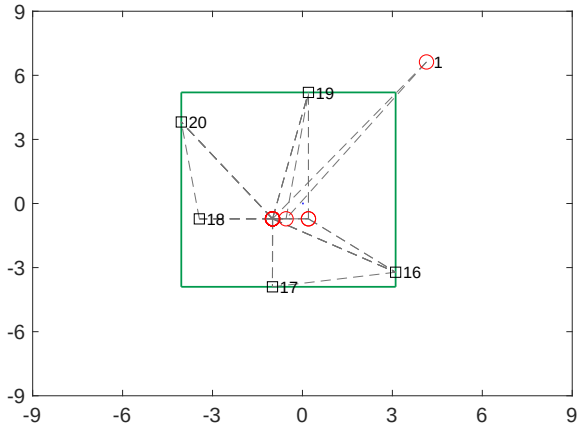
In this paper we considered the containment control problem in a multi-agent system with the presence of malicious agents which attempt to disrupt the emergent behavior of the network by grabbing follower agents away from the leaders. We considered a local interaction protocol based on sign functions which is shown to provide robustness against malicious agents. The considered protocol is implemented by the agents independently on each coordinate axis, therefore it requires the agents to have a common coordinate system, for instance a common heading provided by a compass, and being able to measure bearing angles between them and their neighbors in their local inertial frame. Future work will be mainly focused on the development of a local interaction protocol that: i) does not require a common coordinate frame to achieve robustness against malicious agents, and ii) can consider the presence of dynamical leaders.

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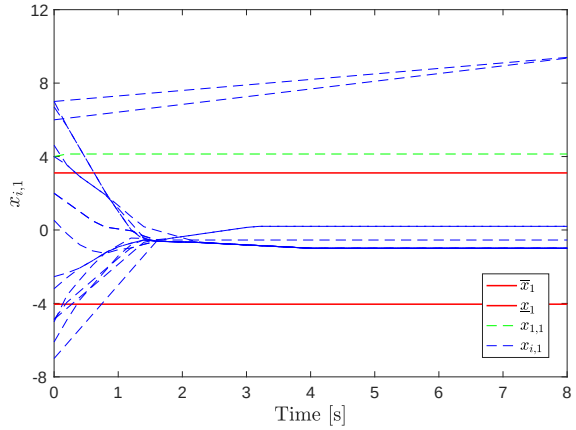
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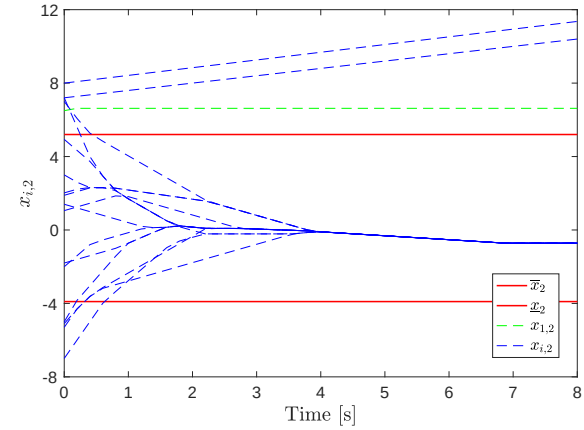
(a) Position of the team of agents at initial time $t = 0$ s.



(b) Position of the team of agents at final time $t = 8$ s.



(c) Evolution of the first component $x_{i,1}$ of the agents with respect to the limits $\overline{x}_1$ and $\underline{x}_1$ of the containment section $\mathcal{C}_1$.



(d) Evolution of the second component $x_{i,2}$ of the agents with respect to the limits $\overline{x}_2$ and $\underline{x}_2$ of the containment section $\mathcal{C}_2$.

Fig. 3. A multi-agent system composed of 15 followers, 5 leaders, 2 malicious agents with $\alpha = 1$ and graph $(4, \mathcal{V}_{f\ell})$-*connected*.

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